1,2,...: Counting to infinity

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- Simply pair each nut with a bolt and hope that they simultaneously run out!

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- Is it true that $A \subset B \Longrightarrow |A| < |B|$?

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- Remark: $|\mathbb{P}| = |\mathbb{N}|$
- Mild shock: By induction: $n\aleph_0 = \aleph_0$ for any n i.e. $(\aleph_0 + \aleph_0 + ... = \aleph_0)$

$$\begin{matrix} p_1 & 2p_1 & 3p_1 & \ldots \\ p_2 & 2p_2 & 3p_2 & \ldots \\ p_3 & 2p_3 & 3p_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

Countable to continuum

- $|\mathbb{Q}| = |\mathbb{N}|$ since

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$$f(x) = \begin{cases} x & \text{if } x \notin \{0, 1, 1/2, 1/3, 1/4 \dots\} \\ \frac{1}{n+2} & \text{if } x = 1/n \\ 1/2 & \text{if } x = 0 \end{cases}$$

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- $|(-1,1)| = |(0,1)|$ with $f : (-1,1) \longrightarrow (0,1)$ such that $x \longmapsto \frac{1}{2}(x+1)$

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then, $y = 0.b_1 b_2 \ldots \in (0,1)$ with $b_k \neq a_{kk}$ (that is, $\aleph_0 < \mathfrak{c}$. By CH, $2^{\aleph_0} = \mathfrak{c}$)

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- That is, there is a bijection from sequence of real numbers to the set of g's
- How many functions of type g are there? Basically count functions of type $h : \mathbb{N} \longrightarrow \{0, 1\}$. Answer: $2^{\aleph_0} = \mathfrak{c}$