Variational Bicomplex and Takens' Theorem

Advanced Topics Exam

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Outline

- Introduction
- Mathematical background
- Mathematical apparatus
- Theorem
- Application
- Proof
- Future Work
- References

Introduction

$$\begin{array}{ccc} \{\text{Vector Fields}\} & \longrightarrow & \{\text{Lagrangian}\} \\ & \swarrow & \\ \{\text{Diff Eqs}\} & \longrightarrow & \{\text{Diff Ops}\} \end{array}$$

$x : \mathbb{R} \longrightarrow \mathbb{E}^n$
$L = \frac{m}{2} \lvert \dot{x} \rvert^2 \, dt = \frac{m}{2} \langle \dot{x}, \dot{x} \rangle \, dt$ and $S = \int L$ $\delta L = m \langle \dot{x}, \delta \dot{x} \rangle \, dt = -m \langle \delta x, \ddot{x} \rangle \, dt + d \{ m \langle \dot{x}, \delta x \rangle \}$

Introduction

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$$\swarrow$$

$$\{\text{Diff Eqs}\} \longrightarrow \{\text{Diff Ops}\}$$

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x is a section of $\mathbb{E}^n \longrightarrow \mathbb{R}$
$L \in \Omega^n(M)$ is a volume form, depending only on first derivative of x
What of the Euler-Lagrange Equation?

Jet Bundles

- Let $E \longrightarrow M$ be a submersion and $s_1, s_2 \in \Gamma_{\text{loc}}(M, E)$ at $x \in M$. Define $s_1 \sim_k s_1 \iff$ partial derivatives agree upto k. Let $j^k(s)(x)$ be the resulting equivalence class.
- The **jet bundle** $\pi^k : J^k \longrightarrow M$ is fibered by $\left(\pi^k\right)^{-1}(x) = j^k(s)(x)$

Jet Bundles

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For $k \leq \ell$, define $\pi^k_\ell : J^\ell(E) \longrightarrow J^k(E)$ as $\pi^k_\ell\left(j^\ell(s)(x)\right) = j^k(s)(x)$

$$J^\infty(E) = \varprojlim J^k(E) = \varprojlim\left(\ldots \longrightarrow J^2(E) \longrightarrow J^1(E) \longrightarrow J^0(E) = E\right)$$

Variational Bicomplex

- $\Omega^k(J^\infty(E)) = \bigoplus_{p+q=k} \Omega^{p,q}(J^\infty(E))$ where splitting is induced by contact ideal generated by $\theta^\alpha_I = du^\alpha_I - u^\alpha_{I\cup\{i\}}dx^i$ where $\left(x^i, u^\alpha, u^\alpha_I\right)$ are coordinates on $J^\infty(E)$

- Let $\mathcal{F} = \Gamma(M, E)$ be sections of submersion $E \longrightarrow M^n$
- $\mathbf{ev} : \mathcal{F} \times M \xrightarrow{(j^\infty, id_M)} \Gamma(J^\infty(E), M) \times M \xrightarrow{ev} J^\infty(E)$

$$\boxed{\Omega^{p,q}_{\text{loc}}(\mathcal{F} \times M) := \mathbf{ev}^*\left(\Omega^{p,q}(J^\infty(E))\right)}$$

$$\begin{array}{ccccccc}
\Omega^{0,|0|}_{\text{loc}}(\mathcal{F}\times M) & \xrightarrow{\delta} & \Omega^{1,|0|}_{\text{loc}}(\mathcal{F}\times M) & \xrightarrow{\delta} & \Omega^{2,|0|}_{\text{loc}}(\mathcal{F}\times M) & \xrightarrow{\delta} & \dots \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
\Omega^{0,|-1|}_{\text{loc}}(\mathcal{F}\times M) & \xrightarrow{\delta} & \Omega^{1,|-1|}_{\text{loc}}(\mathcal{F}\times M) & \xrightarrow{\delta} & \Omega^{2,|-1|}_{\text{loc}}(\mathcal{F}\times M) & \xrightarrow{\delta} & \dots \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
\vdots & & \vdots & & \vdots & & \\
\Omega^{0,|-n|}_{\text{loc}}(\mathcal{F}\times M) & \xrightarrow{\delta} & \Omega^{1,|-n|}_{\text{loc}}(\mathcal{F}\times M) & \xrightarrow{\delta} & \Omega^{2,|-n|}_{\text{loc}}(\mathcal{F}\times M) & \xrightarrow{\delta} & \dots
\end{array}$$

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Formally...

- A field $\phi \in \mathcal{F} = \Gamma(M, E)$. Typically, $E \longrightarrow M$ is a submersion.
- $L \in \Omega^{0,|0|}_{\text{loc}}(\mathcal{F} \times M)$ where, for some k, L depends only on k-jet of $\phi \in \mathcal{F}$ and $\xi_1, \xi_2, ..., \xi_p \in T_\phi \mathcal{F}$
- $ELE := \delta L \mod \text{im} d = 0$ and $S : \Gamma(M, E) \longrightarrow \mathbb{R}$
- Phase space $\mathcal{M} = \{\phi \in \Gamma(M, E) : ELE(\phi) = 0\}$
- Conserved current $\eta \in \Omega^{0,|-1|}_{\text{loc}}(\mathcal{F} \times M)$ such that $d\eta = 0 \mod ELE$
- $\gamma \in \Omega^{1,|-1|}_{\text{loc}}$ is variational 1-form (Cartan form) such that $\delta\gamma$ is a local symplectic form

$$\begin{array}{ccccccc}
\Omega^{0,|0|}(\mathcal{F} \times M) & \xrightarrow{\delta} & \Omega^{1,|0|}(\mathcal{F} \times M) & \xrightarrow{\delta} & \Omega^{2,|0|}(\mathcal{F} \times M) & \xrightarrow{\delta} & \dots \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
\Omega^{0,|-1|}(\mathcal{F} \times M) & \xrightarrow{\delta} & \Omega^{1,|-1|}(\mathcal{F} \times M) & \xrightarrow{\delta} & \Omega^{2,|-1|}(\mathcal{F} \times M) & \xrightarrow{\delta} & \dots \\
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\vdots & & \vdots & & \vdots & &
\end{array}$$

Takens' Theorem

Theorem (Takens)

For $p > 0$, the complex $\left(\Omega_{\text{loc}}^{p,|\bullet|}(\mathcal{F} \times M), d\right)$ of local (twisted) forms is exact, except in the top degree $|\bullet| = 0$.

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For $p = 1$, consider $\delta L = D\mathcal{L} - d\gamma$ with $\gamma \in \Omega_{\text{loc}}^{1,|-1|}$ and $L \in \Omega_{\text{loc}}^{0,|0|}$.

Here, $\mathcal{L} = L + \gamma$ is the total Lagrangian, $\delta L =$ non-exact source $-$ exact form

Toy model

Let $E \longrightarrow M$ be an (oriented!) line bundle and

$$L = a \wedge |da| = a \wedge da$$

be the Lagrangian density with $a \in \Omega^{|-(n-1)|}(M)$

$$\delta L = \delta a \wedge da + a \wedge \delta da = \delta a \wedge da - a \wedge d\delta a.$$

Let $\gamma = a \wedge \delta a$.

$$d\gamma = da \wedge \delta a - a \wedge d\delta a$$

and

$$\begin{aligned} \delta L &= \delta a \wedge da - a \wedge d\delta a \pm (da \wedge \delta a) \\ &= \delta a \wedge da + d\gamma - da \wedge \delta a = \underbrace{2\delta a \wedge da}_{D\mathcal{L}} + d\gamma \end{aligned}$$

The symplectic form is $\delta\gamma = \delta a \wedge \delta a$

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Proof Sketch

Strategy

1. Represent different variations of sections as vector bundles
2. Go local & grade the differentials from multi-indices to single index
3. Reduce the problem to dimension 1
4. Apply a partition of unity argument

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1. Represent different variations of sections as vector bundles
2. Go local & grade the differentials from multi-indices to single index
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4. Apply a partition of unity argument

Let $E \longrightarrow M$ be a submersion, $V_i \longrightarrow E$ be vector bundles for $i = 1, ..., p$, $V = \times_E V_i$ where, $\mathcal{V}_\phi = \Gamma(M, \phi^* V)$. Need to show that the subcomplex $\left(\Omega^{0,\bullet}_{loc,mult}(\mathcal{V}_\phi \times M), d\right)$ is exact except in the top degree $\bullet = 0$.

The complex consists of forms $\alpha(\phi, \xi_1, ..., \xi_p)$ which depend on k-jet of ϕ and on sections ξ_i of $\phi^* V_i$. Moreover, α is $\mathbb{R}$-multilinear in ξ_i

Proof Sketch (Cont.)

Lemma 1

Locally, the chain complex is

$$\bigotimes_{i=1}^{p} \bigotimes_{|n_i|\leq k} \pi^{(k)^*} Sym^{|n_i|}(TM) \otimes \pi^{(k)^*}(\Omega^q(M)) \longrightarrow \bigotimes_{i=1}^{p} \bigotimes_{|n_i|\leq k+1} \pi^{(k+1)^*} Sym^{|n_i|}(TM) \otimes \pi^{(k+1)^*}\left(\Omega^{q+1}(M)\right)$$

with differential $s \otimes \zeta \mapsto \left(\sum_{\ell} \left(\sum_{i=1}^{p} 1 \otimes ... \otimes (e_\ell \text{ at the } i\text{th place}) \otimes ... \otimes 1\right).s\right) \otimes e^{\ell} \wedge \zeta$

Proof

$$\alpha(\phi, \xi_1, ..., \xi_p) = \sum \alpha_{n_1,...,n_p}(\phi)\, \partial^{n_1}\xi_1 ... \partial^{n_p}\xi_p$$

$$N = \sum_{i=1}^{p} |n_i| \text{ and } Gr^F\left(\Omega^{0,q}_{loc,mult}\right) = \bigoplus_{N=0} F_{N+1}/F_N := \bigoplus_{N=0} Gr^q_N$$

where $F_0 \subseteq F_1 \subseteq ... \subseteq F_N \subseteq F_{N+1} \subseteq ...$

Proof Sketch (Cont.)

Next step:

$$\bigotimes_{1}^{p} Sym^{*}\left(T_{m_0}M\right) \otimes \Lambda^{q}\left(T_{m_0}^{\vee}M\right) \cong Sym^{*}\left(S\right) \otimes \overbrace{Sym^{*}\left(T_{m_0}M\right) \otimes \Lambda^{q}\left(T_{m_0}^{\vee}M\right)}^{\text{degree } q}$$

$$\text{with } d\left(s \otimes \zeta\right) = \sum_{\ell} \left(\left(e_{\ell}.s\right) \otimes \left(e^{\ell} \wedge \zeta\right)\right)$$

Proof Sketch (Cont.)

Next step:

$$\bigotimes_{1}^{p} Sym^{*}\left(T_{m_0}M\right) \otimes \Lambda^{q}\left(T_{m_0}^{\vee}M\right) \cong Sym^{*}(S) \otimes \overbrace{Sym^{*}\left(T_{m_0}M\right) \otimes \Lambda^{q}\left(T_{m_0}^{\vee}M\right)}^{\text{degree } q}$$

$$\text{with } d\left(s \otimes \zeta\right) = \sum_{\ell} \left(\left(e_{\ell}.s\right) \otimes \left(e^{\ell} \wedge \zeta\right)\right)$$

Lemma 2

For $T_{m_0}M = A \oplus B$,

$$Sym^{*}(A \oplus B) \otimes \Lambda^{q}(A^{\vee} \oplus B^{\vee}) \cong \left(Sym^{*}(A) \otimes \Lambda^{\bullet}(A^{\vee})\right) \otimes \left(Sym^{*}(B) \otimes \Lambda^{\bullet}(B^{\vee})\right)$$

Proof Sketch (Cont.)

$$0 \longrightarrow K[X] \cong K[X] \otimes K \xrightarrow{m_X} K[X] \otimes K^\vee \cong K[X] \longrightarrow 0$$

where m_X is multiplication by X

$$H^0 = \ker m_X / \{0\} \cong \{0\} \text{ and } H^1 = \mathbb{R}[X] / m_X(\mathbb{R}[X]) \cong \mathbb{R} \quad \square$$

Future work

Gluing of multivalued functionals and associated Lagrangians [Ald02] and Lie algebroids replacing jet bundles [GG20]
Consider $M = \{U_i\}$ and where ... $\substack{\rightarrow\\ \rightarrow\\ \rightarrow} \sqcup U_{ij} \rightrightarrows \sqcup U_i \longrightarrow M$ via

$$\Omega^{p,|-q|,r}(N\mathcal{U}) := \bigoplus_{i_0,\ldots,i_r} \Omega^{p,|-q|}(U_{i_0,\ldots,i_r}) \text{ with } \check{\delta} : \Omega^{p,|-q|,r}(N\mathcal{U}) \longrightarrow \Omega^{p,|-q|,r+1}(N\mathcal{U})$$

$$\text{as } \left(\check{\delta}\omega\right)_{i_0 i_1 \ldots i_r} = \sum_{k=0}^{r} (-1)^k \, \omega|_{i_0 i_1 \ldots \widehat{i_k} \ldots i_r}$$

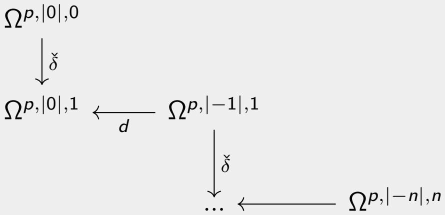
- $L_i = a_i \wedge da_i$ for $a_i \in \Omega^{p,|-(n-2)|,1}(U_i)$
- $\check{\delta} a_{ij} = a_j - a_i = db_{ij}$ for $b_{ij} \in \Omega^{p,|-(n-1)|,2}(U_{ij})$ (well-defined field strength)
- $\check{\delta} b_{ijk} = b_{jk} - b_{ik} + b_{ij} = df_{ijk}$ for $f_{ijk} \in \Omega^{p,|-n|,3}(U_{ijk})$
- $\check{\delta} f_{ijkl} = f_{jkl} - f_{ikl} + f_{ijl} - f_{ijk} = 0$

Future work (Cotd.)

- $L_j - L_i = (a_j - a_i) \wedge da_i = db_{ij} \wedge da_i = d\left(b_{ij} \wedge da_i\right) = L_{ij}$ where $b_{ij} \wedge da_i \in \Omega^{|-(n-4)|,2}\left(U_{ij}\right)$
- $L_{jk} - L_{ik} + L_{ij} = b_{jk} \wedge da_k - b_{ik} \wedge da_k + b_{ij} \wedge da_j = d\left(f_{ijk} \wedge da_i\right)$ where $f_{ijk} \wedge da_i \in \Omega^{|-(n-3)|,3}\left(U_{ijk}\right)$

What if $\delta a_i = \delta a_j$?
Then $\delta L_i = 2\delta a_i \wedge da_i + \left(d + \check{\delta}\right)\gamma$

Or $\delta a_i - \delta a_j = d\delta b_{ij}$?
$\delta L_i = f\left(\delta a_i, da_i, \delta b_i, db_i\right) + \left(d + \check{\delta}\right)\gamma$
with $f\left(\delta a_i, da_i, \delta b_i, db_i\right) \in \Omega^{1,|0|,1}\left(U_i\right)$



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