Operator Algebras and the Foundations of Quantum Mechanics

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Natura non facit saltus - Liebniz

Recap: issues

- $\frac{(0,1)+(1,0)}{\sqrt{2}} \equiv \frac{(0,1)-(1,0)}{\sqrt{2}}$
- Observables may be unbounded (some have empty spectrum)
- $\mathcal{H}$ vs $B(0,1)$
- t is intrinsic.
- Hilbert spaces vs Semi-norm spaces
- Why $\mathbb{C}$? [4][3]
- Why linear operators? [1]
- Why separable?[2] (uncountable eigenvectors)
- Why associative law? [5]

von Neumann

I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more. After all, Hilbert space was obtained by generalising Euclidean space, footing on the principle of 'conserving the validity of all formal rules'. Now we begin to believe that it is not the vectors which matter, but the lattice of all linear (closed) subspaces. Because: 1) The vectors ought to represent the physical states, but they do it redundantly, up to a complex factor, only 2) and besides, the states are merely a derived notion, the primitive (phenomenologically given) notion being the qualities which correspond to the linear closed subspaces [3].

Outline

- Our focus: skew fields $\mathbb{K}$ and seminorm spaces

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Outcomes: Riesz Representation Theorem without completeness

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Outcomes: Adjoint of multivalued operators are single valued

Algebraic Decomposition of Vectors: Basis

- Vectors $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n \in X$ of the form $\sum \alpha_{ij}\mathbf{x}_i\mathbf{x}_j$ will be called a **multiplicative linear combination** and will be **multiplicatively linearly independent** if $\sum \alpha_{ij}\mathbf{x}_i\mathbf{x}_j = 0$ implies $\alpha_{ij} = 0$.

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Example

Instead of $1, i, j, k$ as Hamel basis for the quaternion algebra $\mathbb{H}$ over $\mathbb{R}$, we can have i, j because then $ij = k$ and $i^2 = j^2 = -1$ generates the quaternions. i, j are multiplicative-linearly independent.

Algebraic Decomposition of Vectors: Basis

Lemma

B is a basis if and only if B is minimal. That is, deletion of any element from B, except $\mathbf{0}$ if $\mathbf{0} \in B$, does not form a basis

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Proof.

Letting $\mathbf{x}_i\mathbf{x}_j = \mathbf{x}_l$ in "$\sum \alpha_{ij}\mathbf{x}_i\mathbf{x}_j = 0$ implies $\alpha_{ij} = 0$ for $1 \leq i, j \leq k$" implies linear independence □

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Lemma

Every $\mathbb{K}$-algebra X possesses a m-Hamel basis

Algebraic Decomposition of Vectors: Basis

Theorem

A totally ordered division $\mathbb{K}$-algebra X over a skew field is $\mathbb{K}$.

Proof.

$f : X \longrightarrow \mathbb{F}$ (non-trivial, positive)
$\forall \alpha \in \mathbb{F}$, $\alpha \mathbf{e} \in X$.
$f(\mathbf{x}) = f(\mathbf{y}) \implies \sum \left(\alpha_{ij} - \beta_{ij}\right) f(\mathbf{v}_i) f(\mathbf{v}_j) = 0$ where $\mathbf{x} = \sum \alpha_{ij} \mathbf{v}_i \mathbf{v}_j$ and $\mathbf{y} = \sum \beta_{ij} \mathbf{v}_i \mathbf{v}_j$ are positive elements. Let $a_{ij} = \left(\alpha_{ij} - \beta_{ij}\right) f(\mathbf{v}_i) f(\mathbf{v}_j)$. Then, $\sum a_{ij} = 0 \implies a_{ij} = 0$. Thus, $\left(\alpha_{ij} - \beta_{ij}\right) = 0$, $f(\mathbf{v}_i) = 0$ or $f(\mathbf{v}_j) = 0$. Hence $\alpha_{ij} = \beta_{ij}$. For negative vectors, let $g : X \longrightarrow \mathbb{F}$ such that $g = -f$. □

Mappings

Definition

Let X be a vector space over $\mathbb{F}$ and Y be vector space over $\mathbb{K}$ and let $\phi : \mathbb{F} \longrightarrow \mathbb{K}$ be a homomorphism. Then, an operator $T : X \longrightarrow Y$ is a ϕ-**vector space homomorphism** between X and Y if for all $\mathbf{x}, \mathbf{y} \in X$ and scalars $\alpha \in \mathbb{F}$, $T(\alpha\mathbf{x} + \beta\mathbf{y}) = \phi(\alpha) T(\mathbf{x}) + \phi(\beta) T(\mathbf{y})$. T is an **isomorphism** if T and ϕ are bijective. A ϕ-**algebra homomorphism** is of the form $T((\alpha\mathbf{x})(\beta\mathbf{y})) = T(\alpha\beta\mathbf{xy}) = \phi(\alpha\beta) T(\mathbf{x}) T(\mathbf{y})$, which we shall call an **isomorphism** if ϕ and T are bijective.

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Definition

$T = \{(x, z) : x \in V, z \in W\}$ is a relation, then
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Lemma

Preservation of multiplicative linear dependence if T is injective (not ϕ)

Axioms for seminorm space N

- $\boxed{\|\mathbf{x}\| = 0 \implies \mathbf{x} = 0 \text{ (non-degeneracy)}}$
- $\|\alpha\mathbf{x}\| = |\alpha| \, \|\mathbf{x}\|$ for all $\alpha \in \mathbb{K}$, $\forall \mathbf{x} \in N$ (homogeneity)
- $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for arbitrary $\mathbf{x}, \mathbf{y} \in N$ or $\|\mathbf{x}+\mathbf{y}\| \leq \max\left(\|\mathbf{x}\|, \|\mathbf{y}\|\right)$
- Seminorm from underlying field: $\|\mathbf{x}\| := |g(\mathbf{x})|$
- Outcomes: $\|\mathbf{0}\| = 0$, $\|\mathbf{x}\| = \|-\mathbf{x}\|$ and $\|\mathbf{x}\| \geq 0$
- Norm: N/W where W =set $\mathbf{v}$ s.t. $\|\mathbf{v}\| = 0$
- $\|\mathbf{x}^2\| = \|\mathbf{x}\|^2 \implies \|\mathbf{xy}\| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$[4] $\implies \|\mathbf{e}\| \geq 1$

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- $\left\|\mathbf{x}^2\right\| = \|\mathbf{x}\|^2 \implies \|\mathbf{xy}\| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|[4] \implies \|\mathbf{e}\| \geq 1$
- Axiom of choice!

Sesquilinear forms

Definition

Let X be a vector space over $\mathbb{K}$. A f-**sesquilinear 2-form** is a function $\varphi : X \times X \longrightarrow \mathbb{K}$ such that $\forall \alpha \in \mathbb{K}$ and $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X$

- $\varphi(\mathbf{x} + \mathbf{y}, \mathbf{z}) = \varphi(\mathbf{x}, \mathbf{z}) + \varphi(\mathbf{y}, \mathbf{z})$

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- $\varphi(\mathbf{x} + \mathbf{y}, \mathbf{z}) = \varphi(\mathbf{x}, \mathbf{z}) + \varphi(\mathbf{y}, \mathbf{z})$
- $\varphi(\mathbf{x}, \mathbf{y} + \mathbf{z}) = \varphi(\mathbf{x}, \mathbf{y}) + \varphi(\mathbf{x}, \mathbf{z})$

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- $\varphi(\mathbf{x}, \alpha\mathbf{y}) = \varphi(\mathbf{x}, \mathbf{y})\,\alpha$
- $\varphi(\alpha\mathbf{x}, \mathbf{y}) = f(\alpha)\,\varphi(\mathbf{x}, \mathbf{y})$ where $f : \mathbb{K} \longrightarrow \mathbb{K}$ is an involutive anti-automorphism.

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- $\varphi(\mathbf{x}+\mathbf{y}, \mathbf{z}) = \varphi(\mathbf{x}, \mathbf{z}) + \varphi(\mathbf{y}, \mathbf{z})$
- $\varphi(\mathbf{x}, \mathbf{y}+\mathbf{z}) = \varphi(\mathbf{x}, \mathbf{y}) + \varphi(\mathbf{x}, \mathbf{z})$
- $\varphi(\mathbf{x}, \alpha\mathbf{y}) = \varphi(\mathbf{x}, \mathbf{y})\, \alpha$
- $\varphi(\alpha\mathbf{x}, \mathbf{y}) = f(\alpha)\, \varphi(\mathbf{x}, \mathbf{y})$ where $f : \mathbb{K} \longrightarrow \mathbb{K}$ is an involutive anti-automorphism.

- Outcomes: $\boxed{\varphi(\mathbf{0}, \mathbf{y}) = \varphi(\mathbf{x}, \mathbf{0}) = 0}$, $\boxed{\text{char}\mathbb{K} = 2 \text{ implies } \varphi(\mathbf{v}, \mathbf{v}) = 0 \iff \varphi(\mathbf{v}, \mathbf{w}) = -\varphi(\mathbf{w}, \mathbf{v})}$, $\boxed{\varphi(\mathbf{x}, \mathbf{y}) = f(\varphi(\mathbf{y}, \mathbf{x})) \iff \varphi(\mathbf{x}, \mathbf{x}) \in \mathbb{R}}$ and $\boxed{\varphi(\mathbf{x}, \mathbf{x})\varphi(\mathbf{y}, \mathbf{y}) \geq \varphi(\mathbf{x}, \mathbf{y})\varphi(\mathbf{y}, \mathbf{x}) \text{ if } \varphi(\mathbf{x}, \mathbf{x}) \geq 0}$

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Lemma

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Proof.

$\|\alpha\mathbf{x}\|^2 = |\alpha|^2 \|\mathbf{x}\|^2$ and $\|\mathbf{x}+\mathbf{y}\|^2$
$\leq |\varphi(\mathbf{x},\mathbf{x})| + |\varphi(\mathbf{x},\mathbf{y})| + |\varphi(\mathbf{y},\mathbf{x})| + |\varphi(\mathbf{y},\mathbf{y})| \leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2$
$\leq \max(|\varphi(\mathbf{x},\mathbf{x})|, |\varphi(\mathbf{x},\mathbf{y})|, |\varphi(\mathbf{y},\mathbf{x})|, |\varphi(\mathbf{y},\mathbf{y})|)$. Now, if
$\varphi(\mathbf{x},\mathbf{y}) = a + b$ for $a, b \in \mathbb{K}$ for $f(a) = a$ and $f(b) \neq b$, then
$|a|,|b| \leq \|\mathbf{x}\| \|\mathbf{y}\|$ [1] $\implies \max\{|a|, |b|\} \leq \|\mathbf{x}\| \|\mathbf{y}\|$ so that
$\max(|\varphi(\mathbf{x},\mathbf{x})|, |\varphi(\mathbf{y},\mathbf{x})|, |\varphi(\mathbf{y},\mathbf{y})|) = \max\{\|\mathbf{x}\|, \|\mathbf{y}\|\}$
If $\mathbf{x} \neq 0$ implies $\varphi(\mathbf{x},\mathbf{x}) > 0$, then N1 □

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$\varphi(\mathbf{x},\mathbf{x}) := \|\mathbf{x}\|^2$ *for* φ*=Hermitian and* $|f(\alpha)| = |\alpha|$

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$\leq \max(|\varphi(\mathbf{x},\mathbf{x})|, |\varphi(\mathbf{x},\mathbf{y})|, |\varphi(\mathbf{y},\mathbf{x})|, |\varphi(\mathbf{y},\mathbf{y})|)$. Now, if
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$|a|,|b| \leq \|\mathbf{x}\| \|\mathbf{y}\|$ [1] $\implies \max\{|a|, |b|\} \leq \|\mathbf{x}\| \|\mathbf{y}\|$ so that
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If $\mathbf{x} \neq 0$ implies $\varphi(\mathbf{x},\mathbf{x}) > 0$, then N1 □

- $|\varphi(\mathbf{x},\mathbf{y})| \leq m \|\mathbf{x}\| \|\mathbf{y}\| \implies \varphi(\mathbf{x}_n,\mathbf{y}_n) \longrightarrow \varphi(\mathbf{x},\mathbf{y})$

Closed subspaces and associated algebra[2]

- $A \longmapsto A^{\perp\perp} \implies A \subseteq A^{\perp\perp}$, $A \subseteq B \implies A^{\perp\perp} \subseteq B^{\perp\perp}$ and $A^{\perp\perp\perp\perp} = A^{\perp\perp}$

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Theorem

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Proof.

Assume there exists a closed B such that $A \subset B \subseteq A^{\perp\perp}$. Then, $B = B^{\perp\perp}$ and $A \subset B^{\perp\perp} \subseteq A^{\perp\perp}$ so that $B^{\perp} \subset A^{\perp}$ and $A^{\perp\perp\perp} = A^{\perp} \subseteq B^{\perp}$ and hence $B^{\perp\perp} = A^{\perp\perp}$. □

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Theorem

A closed relation ($T = T^{\perp\perp}$) T is linear

Closed subspaces and associated algebra[2]

Proof.

T is a subspace of $X \oplus X$. Plus $T(\alpha x) = f(\alpha)\, T(x)$ if
$\alpha (x, y) := (\alpha x, f(\alpha)\, y)$ □

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- $T + S = \{(x, y) : y = s + t \text{ for } s \in \mathcal{R}(S), t \in \mathcal{R}(T)\}$

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- $RS + RT \subseteq R(S + T)$. Converse holds if $\mathcal{D}(R) = X$

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- $RS + RT \subseteq R(S + T)$. Converse holds if $\mathcal{D}(R) = X$
- $(S + T)R \subseteq SR + TR$. Converse holds if R is single-valued.

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Theorem

If T is closed, then $\ker(T)$ *is closed*

- $T + S = \{(x, y) : y = s + t \text{ for } s \in \mathcal{R}(S), t \in \mathcal{R}(T)\}$
- $T \circ S = TS = \{(x, z) : (x, y) \in S \text{ and } (y, z) \in T \text{ for some } y\}$
- $O = \{(x, 0)\}$, $I = \{(x, x)\}$ and $\lambda T = \{(x, \lambda y) : (x, y) \in T\}$
- $RS + RT \subseteq R(S + T)$. Converse holds if $\mathcal{D}(R) = X$
- $(S + T)R \subseteq SR + TR$. Converse holds if R is single-valued.
- $\ker T = \ker S$ and $\mathcal{R}(S) = \mathcal{R}(T)$, then $S \subset T$ implies $S = T$.

Closed subspaces and associated algebra[2]

Theorem

A single-valued, linear adjoint of densely defined $\left(\mathcal{D}\left(T\right)^{\perp\perp} = X\right)$ *relation* T *will always exist*

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Proof.

$U : X \times X \longrightarrow X \times X$ by $U\left(x, y\right) := \left(-y, x\right)$. Well-defined+bijective.
$\Phi\left(z, w\right) := \left(\varphi \oplus \varphi\right)\left(z, w\right)_{X \times X} = \varphi\left(z_1, w_1\right) + \varphi\left(z_2, w_2\right)$
$\Longrightarrow \Phi\left(U\left(z\right), w\right) = \Phi\left(z, U^{-1}\left(w\right)\right)$
For $M \subseteq X \times X$, $T^* = U\left(M^{\perp}\right) = U\left(M\right)^{\perp}$
$\varphi\left(Tx, y\right) = \varphi\left(x, T^* w\right)$ for $\left(x, z\right) \in T$ and $\left(y, w\right) \in T^*$ □

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Outcomes $\mathcal{D}(T)^{\perp\perp} = E \iff T^*$ is single-valued

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- $\|T\| = \sup\limits_{\|\mathbf{x}\|\neq 0} \frac{\|T\mathbf{x}\|}{\|\mathbf{x}\|} \implies \|RT\| \leq \|R\|\,\|T\|$ (care for $\|\alpha T\| = |\phi(\alpha)|\,\|T\|$)

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Theorem

A unital Weak Banach algebra $(X, \|.\|)$ *is a complete subalgebra of* $B_\phi(X)$

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$L_{\mathbf{x}}(\mathbf{y}) := \mathbf{xy}$. Then, $L_{\mathbf{x}} \in B_\phi(X)$. Then, $L : X \longrightarrow B_\phi(X)$ as $L(\mathbf{x}) = L_{\mathbf{x}}$ is a homomorphism and $\|\mathbf{x}\|_o := \|L_{\mathbf{x}}\|$ is equivalent to $\|.\|$ □

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- Proof: $\forall\lambda \in \mathbb{K}$, $\lambda I \in B_\phi(X) \implies g(I) = e$. Consider orthogonal projection operators P and $Q \in B_\phi(X)$ s.t. $\dim P(X) = \dim Q(X)$. Then, $T : P(X) \longrightarrow Q(X)$, a partial isometry such that $P = T^*T$, $Q = TT^*$ so that $PQ = 0 \implies g(Q) = g(P) = 0$. Further, $P + Q = I \implies e = g(I) = g(P) + g(Q) = 0$

Riesz Representation Theorem on Hermitian Spaces[2]

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$\exists$ *cts linear functional* $g : (X, \varphi, \mathbb{K}) \longrightarrow X^*$ *such that* $\mathcal{R}(g) = X'$.

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Proof.

$g_y : X \longrightarrow X^*$ s.t. $g_y(x) = \varphi(y, x)$
(injective+well-define) $\Longrightarrow \mathcal{R}(g) \subseteq X^{'}$. g_y cts since
$\ker g_y = \{ky : k \in \mathbb{K}\}^{\perp}$
Conversely, for $h \in X'$, $h = 0 \implies g_0 = h \implies h \in \mathcal{R}(g)$.
$h \neq 0 \implies \dim h = 1$
$\implies X = \ker h \oplus \{kv : k \in \mathbb{K}\}$. Letting $w = f^{-1}\left(\varphi(v, z)^{-1} h(v)\right) z$
for $z \in \ker h^{\perp}$ and $z \notin \{kv : k \in \mathbb{K}\}^{\perp}$ gives us $h(v) = \varphi(v, w)$.
$X \ni x = x_1 + \alpha v \implies h(x) = \alpha h(v) \implies \varphi(x, w) = \alpha\varphi(v, w) \implies$
$h = g_w$ □

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Kernel of each element of $g(A)$ is splitting.

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Proof.

If y is anisotropic, then $y \notin \{ky : k \in \mathbb{K}\}^{\perp}$ so
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Proof.

If $0 \neq y \in X$ such that $\varphi(y, y) = 0$, then
$\{ky : k \in \mathbb{K}\} \oplus \{ky : k \in \mathbb{K}\}^{\perp} \subset X$ □

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- If a Hermitian space is orthomodular, then $\langle F \rangle = F^{\perp\perp}$ and such sets form atomic ortholattice which is isomorphic to the lattice of closed subspaces of a Hilbert space over an arbitrary Archimedean skew field[6].

Solr's theorem

Theorem

Let $(X, \mathbb{K}, \varphi)$ *be an infinite dimensional orthomodular space over a skew field* $\mathbb{K}$ *which contains an orthonormal system* $(e_i)_{i \in \mathbb{N}}$*. Then* $\mathbb{K}$ *is either* $\mathbb{R}$*,* $\mathbb{C}$ *or* $\mathbb{H}$ *and* $(X, \mathbb{K}, \varphi)$ *is a Hilbert space [4]*

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Proof.

$nx = \langle \sum_{i=0}^{n} e_i \rangle x = 0 \iff \langle \sum_{i=0}^{n} e_i \rangle = 0 \iff n = 0$

$\implies \mathbb{Q} \subset \mathbb{K}$

$\forall (\alpha_i)_{i\in\mathbb{N}^*} \in \mathbb{Q}^{\mathbb{N}^*}$ with $\alpha := \sum_{i=0}^{\infty} \alpha_i^2 \in \mathbb{Q}$, then $\exists x = \sum_{i\in\mathbb{N}^*} \alpha_i e_i \in X$, with $\langle x \rangle = \alpha$

Define $\sum_{i=0}^{\infty} \alpha_i^2 \longmapsto \langle \sum_{i\in\mathbb{N}} \alpha_i e_i \rangle$

This is multiplicative linear function so that $\mathbb{R} \subset \mathbb{K}$

$\implies (\alpha_i)_{i\in\mathbb{N}} \in l_2(\mathbb{R})$ with $\alpha := \sum_{i=0}^{\infty} \alpha_i^2$, $\exists x = \sum_{i\in\mathbb{N}} \alpha_i e_i \in X$ such that $\langle a \rangle = \alpha$ □

Proof.

(cotd.)
Next, $\mathbb{R} \subset Z = \{x \mid xy = yx, \forall y \in \mathbb{K}\} \implies \mathbb{R} = S(\mathbb{K})$ using
$$S \subseteq P := \left\{ \langle x \rangle \mid 0 \neq x = \sum_{i \in \mathbb{N}} \xi_i e_i,\ \xi_i \in \mathbb{R}(\gamma) \forall i \in \mathbb{N} \text{ and } \langle x \rangle \in \mathbb{R}(\gamma) \right\}$$
where $\gamma \in S$
$\lambda \in \mathbb{K} \setminus \mathbb{R} \implies \mathbb{R}(\lambda) \cong \mathbb{C}$
$\lambda \in \mathbb{K} \setminus \mathbb{C} \implies \mathbb{C} + \mathbb{C}\lambda \cong \mathbb{H} \implies$
$\lambda \in \mathbb{K} \setminus \mathbb{H} \implies \mathbb{H} + \mathbb{H}\lambda \cong \mathbb{H}$, contradiction
Hence $X \cong l_2(\mathbb{K})$ and $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ □

Conclusion

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- Non-Archimedean fields

Future Work

- Does there exist a (countable?) eigenbasis decomposition of a non-linear operator on a Hermitian space over a non-Archimedean field?

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- Does there exist a (countable?) eigenbasis decomposition of a non-linear operator on a Hermitian space over a non-Archimedean field?
- Over which non-Archimedean fields are Hermitian spaces orthomodular?

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