Differential Cohomology

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This paper is an extension of [3]. Through-out this paper, we let M be a manifold.

Introduction

Differential Cohomology is a cohomological theory which extends a topological cohomology theory, i.e., a generalized cohomology which satisfies Eilenberg-Steenrod axioms, an example of which is singular cohomology $C^{\bullet}(M, -) = \mathtt{Hom}(Z[S_{\bullet}(M)], -)$. It does this by taking into account geometric data of the underlying topological space. There are instances in which cohomology of the latter is agnostic about details of the former. For example [5], let $P \longrightarrow M$ be a principal $U(1)$-bundle with connection θ. The class $[F]$ of 2-forms in $H^2(M;\mathbb{R})$ in real coefficients (equivalently, de Rahm cohomology $H^2_{dR}(M)$ via the de Rahm theorem) represented by the curvature F of P do not determine the structure of P. Determining P up to isomorphism would require an element of $H^2(M;\mathbb{Z})$, cohomology with integral coefficients. However, the inclusion $H^2(M;\mathbb{Z}) \hookrightarrow H^2(M;\mathbb{R})$ is not guaranteed, thanks to presence of $\mathtt{Ext}$. Thus, knowing $[F]$ with real coefficients does not necessarily give us unique information when the coefficients are restricted to the integers. In this way, two non-isomorphic principal bundles P and P', say two closed manifolds with different genera g and g', can have the same $[F]$. Differential Cohomology aims to recover this loss of information.

The starting point of differential cohomology is a (smooth) manifold M to which one assigns a group, allowing us to view differential cohomology as a functor from $\hat{H} : \mathcal{T}op \longrightarrow \mathcal{G}rp$, coupled with a forgetful functor $\hat{H} \longrightarrow H$, where H is a topological cohomology theory that ignores geometric data of M. The target category of $\hat{H}$ is more refined: $\hat{H}^{\bullet}(M)$ is a topological abelian group. The support for $\hat{H}$ is usually defined for non-negative integers. $\hat{H}$ is called a **smooth refinement** of H. The task of differential cohomology thus is to factor a topological cohomology theory with real coefficients on a manifold M through those with integer coefficients, by sending differential cohomology classes to its curvature differential form. Simultaneously, differential cohomology must also project to its underlying ordinary cohomology class through closed, differential forms. Pictorially, the task loosely is

$$\begin{array}{ccccc} & & H^k(M;\mathbb{Z}) & & \\ & \nearrow & & \searrow & \\ \hat{H}^k(M;\mathbb{Z}) & & & & H^k(M;\mathbb{R}) \\ & \searrow & & \nearrow & \\ & & \Omega^k_{\mathrm{cl}}(M) & & \end{array} \tag{1}$$

How does the k-th member relate with other members of the complex?

Recall that we have

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d} & \Omega^{k-1}(M) & \xrightarrow{d} & \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) & \xrightarrow{d} & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & Z^{k-1} & & B^k & & \Omega^{k+1}_{\mathrm{cl}}(M) & & \end{array}$$

where $B^k = d\left(\Omega^{k-1}(M)\right)$, $Z^{k-1} = \ker\left(\Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M)\right)$ and $\Omega^k_{\text{cl}}(M)$ is the collection of closed k-differential forms. Therefore, we can have the map

$$\Omega^{k-1}(M)/B^{k-1} \longrightarrow \Omega^k_{\text{cl}}(M) \tag{2}$$

since $d\left(\omega + B^{k-1}\right) = d\omega$ is an exact and hence closed form. We also have $B^{k-1} \hookrightarrow Z^{k-1} \hookrightarrow \Omega^{k-1}(M)$ giving rise to the map

$$H^{k-1}(M;\mathbb{R}) \cong H^{k-1}_{dR}(M) = Z^{k-1}/B^{k-1} \longrightarrow \Omega^{k-1}(M)/B^{k-1}. \tag{3}$$

Finally, consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0$$

and apply the covariant functor $\mathtt{Hom}_{\mathbb{Z}}(C_{k-1}(M), -)$ where $C_{k-1}(M) = \mathbb{Z}[S_{k-1}(M)]$ is the free abelian group over singular $k-1$-simplices $S_{k-1}(M) = Map\left(\Delta^{k-1}, M\right)$ where $\Delta^{k-1} := \{(x_0, ..., x_k) : x_0 + ... + x_k = 1, x_i \geqslant 0 \forall i\} \subset \mathbb{R}^k$ is a $k-1$ simplex. Because $C_{k-1}(M)$ is free, $C^{k-1}(M, -) := \mathtt{Hom}(C_{k-1}(M), -)$ is exact and hence we have a short exact sequence

$$0 \longrightarrow \mathtt{Hom}_{\mathbb{Z}}(C_{k-1}(M), \mathbb{Z}) \longrightarrow \mathtt{Hom}_{\mathbb{Z}}(C_{k-1}(M), \mathbb{R}) \longrightarrow \mathtt{Hom}_{\mathbb{Z}}(C_{k-1}(M), \mathbb{R}/\mathbb{Z}) \longrightarrow 0$$

That is, we have a sequence of cochain complexes

$$0 \longrightarrow C^{k-1}(M;\mathbb{Z}) \longrightarrow C^{k-1}(M;\mathbb{R}) \longrightarrow C^{k-1}(M;\mathbb{R}/\mathbb{Z}) \longrightarrow 0$$

By appropriate, multiple uses of the Snake Lemma, we get a connecting homomorphism β, called the Bockstein homomorphism:

$$... \longrightarrow H^{k-1}(M;\mathbb{Z}) \longrightarrow H^{k-1}(M;\mathbb{R}) \longrightarrow H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \xrightarrow{\beta} H^k(M;\mathbb{Z}) \longrightarrow H^k(M;\mathbb{R}) \longrightarrow ... \tag{4}$$

Putting all this information together allows an expansion of 1 as follows:

$$\begin{array}{ccccccc} & & H^{k-1}(M;\mathbb{R}/\mathbb{Z}) & \longrightarrow & H^k(M;\mathbb{Z}) & & \\ & \nearrow & & \searrow \quad \nearrow & & \searrow & \\ H^{k-1}(M;\mathbb{R}) & & & \hat{H}^k(M;\mathbb{Z}) & & & H^k(M;\mathbb{R}) \\ & \searrow & & \nearrow \quad \searrow & & \nearrow & \\ & & \Omega^{k-1}(M)/im(d) & \xrightarrow{d} & \Omega^k_{\text{cl}}(M) & & \end{array} \tag{5}$$

Note that the right half of the above diagram is motivated by geometry, whereas the left half of the diagram is ‘purely homotopy-theoretic’ in nature.

But does such a cohomology exist in nature?

Base case

To motivate this definition, let us compute the low-hanging fruit $\hat{H}^0(M;\mathbb{Z})$ and $\hat{H}^1(M;\mathbb{Z})$. First, recall that $\mathbb{R}/\mathbb{Z}$ is a category, like any group. That is, a 1-category. It is, in fact, 1-truncated and can be seen as sitting inside a 2-category, because there are morphisms between morphisms of this category, which correspond to paths between points on the circle. Viewing $Map\,(M,-)$ as a functor which preserves this level of truncations, we see that $Map\,(M,\mathbb{R}/\mathbb{Z})$ is also 1-truncated. Next, recall that every continuous map is homotopic to a smooth map. Thus, viewing $\mathbb{R}/\mathbb{Z}$ as a manifold, we have $[M,\mathbb{R}/\mathbb{Z}] := \pi_0 Map\,(M,\mathbb{R}/\mathbb{Z}) \cong \pi_0 C^\infty\,(M,\mathbb{R}/\mathbb{Z})$. Because of this homotopy equivalence, we conclude that $C^\infty\,(M,\mathbb{R}/\mathbb{Z})$ is also 1-truncated. Next, recall that $\pi_n\,(\mathbb{R}/\mathbb{Z}) \cong \pi_n\,(U\,(1)) \cong \mathbb{Z}$ for $n = 1$ and is trivial for all other n. That is, $\mathbb{R}/\mathbb{Z}$ is a $K\,(\mathbb{Z},1)$ space. Also recall that for any based CW-complex X, the set $[X, K\,(\mathbb{Z},1)]$ of (based) homotopy classes of (based) maps from X to $K\,(\mathbb{Z},1)$ is defined to be $H^1\,(X,\mathbb{Z})$. Since manifolds have the homotopy-type of CW-complexes, we have $[M,\mathbb{R}/\mathbb{Z}] = [M, K\,(\mathbb{Z},1)] := H^1\,(M,\mathbb{Z})$. Thus, the space $\pi_0\,(C^\infty\,(M,\mathbb{R}/\mathbb{Z}))$ of path components $C^\infty\,(M,\mathbb{R}/\mathbb{Z})$ is isomorphic to $H^1\,(M,\mathbb{Z})$. In particular, we have

$$\pi_0 : C^\infty\,(M,\mathbb{R}/\mathbb{Z}) \twoheadrightarrow H^1\,(M;\mathbb{Z}) \tag{6}$$

But there's more!

$$\pi_1\,(C^\infty\,(M,\mathbb{R}/\mathbb{Z})) \cong \pi_0 \Omega C^\infty\,(M,\mathbb{R}/\mathbb{Z}) \cong \pi_0 Map_*\left(S^1, C^\infty\,(M,\mathbb{R}/\mathbb{Z})\right) \cong \pi_0 Map_*\left(S^1, Map\,(M,\mathbb{R}/\mathbb{Z})\right)$$

Lemma 1 $\pi_0 Map_*\left(\left(S^1,s\right),\left(Map\,(M,\mathbb{R}/\mathbb{Z}),\alpha\right)\right) \cong \pi_0 Map\left(M,\left(Map_*\left(\left(S^1,s\right),(\mathbb{R}/\mathbb{Z},r)\right)\right)\right)$

Proof. This is simply the topolgical version of adjunction via currying. Precision calls for consideration of base points. Two maps are in the same path component if and only if they are homotopic. Thus, on the left hand side, we have $f,g : S^1 \longrightarrow Map\,(M,\mathbb{R}/\mathbb{Z})$ where $f \simeq_s g$. That is, $\exists$ continuous $h : S^1 \times [0,1] \longrightarrow Map\,(M,\mathbb{R}/\mathbb{Z})$ such that $h\,(s,0) = f\,(s)$, $h\,(s,1) = g\,(s)$ and $f\,(s)\,(m) = g\,(s)\,(m) = \alpha\,(m)$ $\forall m \in M$ and $\forall s \in S^1$. On the right, we have $F,G : M \longrightarrow Map_*\left(\left(S^1,s\right),\mathbb{R}/\mathbb{Z},r\right)$ with continuous $H : M \times [0,1] \longrightarrow Map_*\left(\left(S^1,s\right),\mathbb{R}/\mathbb{Z},r\right)$ such that $H\,(m,0) = F\,(m)$, $H\,(m,1) = G\,(m)$ where $G\,(m)\,(s) = r = F\,(m)\,(s)$. These data sets are the same. The choice of the base point in the latter is immaterial, since for $\alpha\,(m)\,, r \in \mathbb{R}/\mathbb{Z}$, we know that there exists a contractible path connecting these two points. ∎

By definition, $Map_*\left(S^1,\mathbb{R}/\mathbb{Z}\right)$ is the loop space $\Omega\,(\mathbb{R}/\mathbb{Z})$. Thus,

$$\pi_1 C^\infty\,(M,\mathbb{R}/\mathbb{Z}) \cong \pi_0 Map\left(M, Map_*\left(S^1,\mathbb{R}/\mathbb{Z}\right)\right) \cong \pi_0 Map\,(M,\Omega\,(\mathbb{R}/\mathbb{Z})) \cong H^0\,(M;\mathbb{Z})\,.$$

In particular, there is a surjective map

$$\pi_1 : C^\infty\,(M,\mathbb{R}/\mathbb{Z}) \twoheadrightarrow H^0\,(M;\mathbb{Z}) \tag{7}$$

We will see later that this a characteristic class map of degree 1.

For our second 'natural' map fitting in the diagram for the base case, we define

$$curv : C^\infty\,(M,\mathbb{R}/\mathbb{Z}) \longrightarrow \Omega^1_{cl}\,(M) \tag{8}$$

by $curv\,(f) = f^*\,(vol)$, where $*$ indicates pullback and vol is the volume element on S^1, say $d\theta$. Because d commutes with pullbacks, $d\,(f^*\,(d\theta)) = f^* d\,(d\theta) = 0$, making $curv\,(f)$ a bonafide closed form, hence the map is well-defined. To find the kernel, note that $f^*\,(d\theta) = df = 0$ by the good-old chain rule, implying that f has to be a locally constant map $f : M \longrightarrow \mathbb{R}/\mathbb{Z}$. That is,

$$\ker curv \cong H^0\,(M;\mathbb{R}/\mathbb{Z})$$

As for the image of the map $curv$, let $i : S^1 \hookrightarrow M$ be an embedding, c be a chain and $f : M \longrightarrow \mathbb{R}/\mathbb{Z} \cong S^1$ be a map. Recall that any element $g \in Map\left(S^1, S^1\right)$ has the property that $g(\theta + 2\pi) - g(\theta) \in \mathbb{Z}$. Thus, for $f \circ i = g$,

$$\int_c f^* d\theta := \int_{S^1} i^* f^* d\theta = \int_{S^1} (f \circ i)^* d\theta = \int_{S^1} g^* d\theta = \int_{S^1} dg = g(2\pi) - g(0) \in \mathbb{Z}$$

and so,

$$im\,(curv) = \left\{\alpha \in \Omega^1_{\text{cl}}(M) : \int_{S^1} \alpha \in \mathbb{Z} \text{ for all embeddings } S^1 \hookrightarrow M\right\}$$

It is time for a definition.

Definition 2 *Let $k \geqslant 0$ be an integer. A closed k-form ω has **integral periods** if, for every smooth k-cycle c in M*

$$\int_c \omega \in \mathbb{Z}$$

We write $\Omega^k_{\mathbb{Z}}(M)$ for collection of k-forms with integral periods. Of course $\Omega^k_{\mathbb{Z}}(M) \subset \Omega^k_{cl}(M)$ is a subgroup, since for $\omega_1, \omega_2 \in \Omega^k_{\mathbb{Z}}(M)$, $\omega_2 - \omega_1 \in \Omega^k_{\mathbb{Z}}(M)$ because

$$\int_c (\omega_2 - \omega_1) = \int_c \omega_2 - \int_c \omega_1 \in \mathbb{Z}$$

Therefore, $im\,(curv) \cong \Omega^1_{\mathbb{Z}}(M)$.

Finally, we also have the natural map

$$p : \Omega^0(M) = C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R}/\mathbb{Z}) \tag{9}$$

given by $f \longmapsto \pi \circ f$, where $\pi : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$. Again, $\ker p = \{f \mid \pi \circ f = 0\} = \{f \mid f \text{ is integer valued}\} \subset C^\infty(M, \mathbb{R})$ consists of smooth, integer-valued functions. Such functions are locally constant. Moreoever, $d(\pi \circ f) = d\pi \circ df = 0$, making $\pi \circ f$ closed. If c is any 0-chain on M, then c is a linear combination of points and so,

$$\int_c \pi \circ f = \sum c_i \int_{pt_i} \pi \circ f = 0$$

That is, $\ker(p) \cong \Omega^0_{\mathbb{Z}}(M)$.

More generally, we have

Lemma 3 *$\omega \in \Omega^k_{\mathbb{Z}}(M)$ if and only if $[\omega] \in H^k_{dR} \cong H^k(M, \mathbb{R})$ lies in the image of the map $H^k(M, \mathbb{Z}) \longrightarrow H^k(M, \mathbb{R})$.*

Proof. ($\implies$)It suffices to write out the de Rham isomorphism at the cochain level, explicitly:

$$I : \Omega^k(M) \longrightarrow C^k(M)$$

is given by $I(\omega)$ where

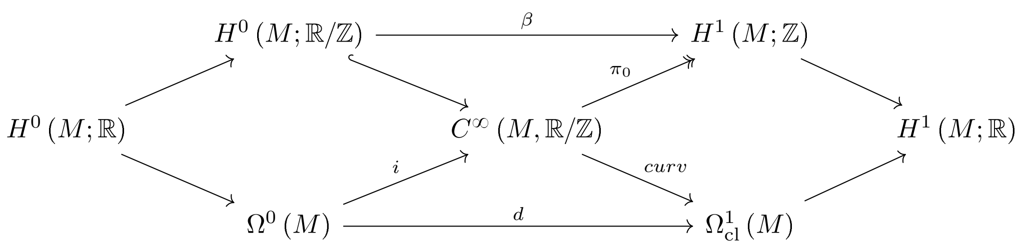
$$I(\omega)_c = \int_c \omega = \int_{[0,1]^k} c^*(\omega) \in \mathbb{Z}$$

Thus, $\omega \in \Omega^k_{\mathbb{Z}}(M) \implies I(\omega)_c \in \mathbb{Z}$ for all k-chains $c \implies [I(\omega)_c] \in \mathbb{Z}$ for all k-chains c

($\Longleftarrow$) Trivial since if $[\omega] = \omega + d\alpha$ is integer-valued, then $\int_c \omega \in \mathbb{Z}$ for $\alpha = 0$. ∎

This is one way in which differential cohomology aims to recover loss of information highlighted in the opening paragraph.

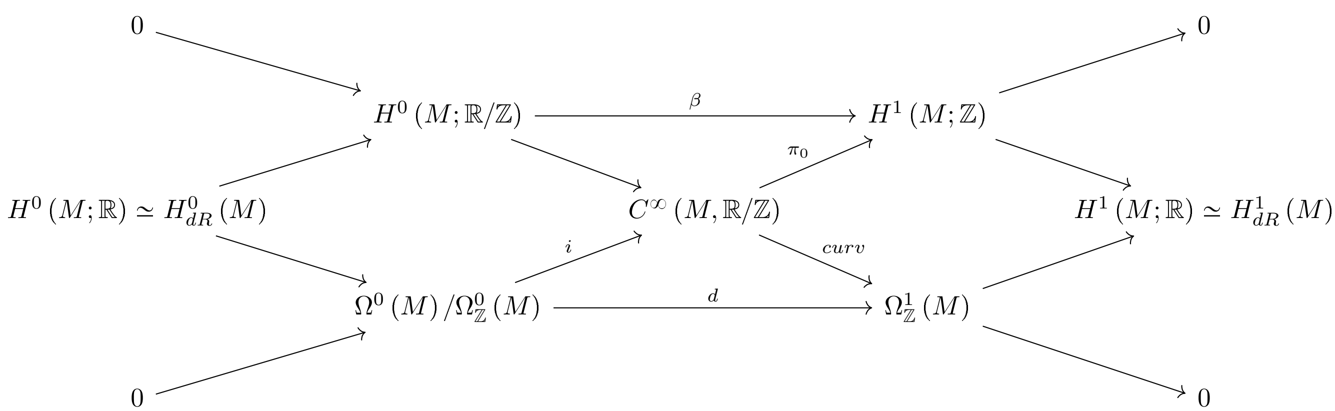
The commutative diagram 5 is now



(10)

where the map i is from 9, π_0 in 6 and β is the Bockstein homomorphism in 4.

Differential cohomology, however, requires the diagonals to be exact. To this end the following modification is made:



We have our diagram 10 for $k = 1$. Observe that the diagram is degenerate for $k = 0$. Therefore, following convention is made: $\hat{H}^0(M;\mathbb{Z}) = H^0(M;\mathbb{Z})$. In particular, the first cohomology is given by the discrete abelian group $\hat{H}^0(M) = Map(M,\mathbb{Z})$.

Background

Differential Cohomology was introduced by Cheeger and Simons (of Chern-Simons fame) in 1973 [1]. The original definition by Cheegar and Simons for differential cohomology constructed $\hat{H}^*(M)$ as a graded ring of differential characters on M. This was built on earlier work by Chern and Simons[2], and later refined by Simons and Sullivan with a more axiomatic approach [8], subsuming earlier approaches and the Deligne cohomology, among others. Now, the more accepted axioms include satisfying the diagram 10, called the "differential cohomology diagram" [7]. An alternate characterization is presented by Hopkins and Singer in [4], with the main result that every generalized cohomology theory admits a differential cohomological refinement (Theorem 3.8 of [6] and Theorem 2.17 of [4]). The paper is motivated by the problem of quantization of the M5-brane by Edward Witten [6].

In fact, differential cohomology also has applications in other areas of physics, and, vice versa, stems from considerations in physics, as well. Electromagnetic fields are differential cocycles in degree 2 in de Rham cohomology, whereas magnetic monopoles are differential cocycles in degree 2 in differential cohomology. More generally, differential cocycles model gauge fields [6].

Machinery of Differential Cohomology

Differential characters model connections on particular bundles. As the name implies, they are characters in particular, hence a homomorphism. The original definition[1] suggested the set of differential k-characters $\hat{H}^k(M;\mathbb{Z})$ as the collection of homomorphisms $\chi : Z_{k-1}(M;\mathbb{Z}) \longrightarrow \mathbb{R}/\mathbb{Z}$ from (smooth) integer-valued k-cycles to the circle group, subject to the condition that

$$\chi(\partial c) = \int_c \omega \bmod \mathbb{Z}$$

for every (smooth) chain $c \in C_k(M;\mathbb{Z})$ and some $\omega \in \Omega^k(M)$. Here, $\partial : C_k \longrightarrow C_{k-1}$ is the boundary map. Existence of $\omega \in \Omega^k_{\mathbb{Z}}(M) \subset \Omega^k(M)$ follows from **Lemma 3**. This gives us a map, called **curvature** of χ, given by the map $curv : \hat{H}^k(M;\mathbb{Z}) \longrightarrow \Omega^k$ defined via $\chi \longmapsto \omega$.

To show that this map is well-defined, we need to show that the ω determined by χ is unique. To this end, let ω_1 and ω_2 be two such differential k-forms with

$$\int_c \omega_1 - \int_c \omega_2 \in \mathbb{Z}$$

That is, for all chains c,

$$\int_c \omega_1 - \omega_2 = 0 \bmod \mathbb{Z}.$$

Thus, $\omega_1 = \omega_2$. The image of $curv$ is $\Omega^k_{\mathbb{Z}}(M)$. To show this, observe that

$$\int_c d\omega = \int_{\partial c} \omega = \chi\left(\partial^2 c\right) = \chi(0) = 0$$

for every chain c. Thus, $d\omega = 0$. To show that $\int_c \omega \in \mathbb{Z}$, or equivalently, $\chi \circ \partial = 0$, it suffices to note that the codomain of ∂ comprises of integer-valued functions.

The characteristic class map $ch : \hat{H}^k(M;\mathbb{Z}) \longrightarrow H^k(M;\mathbb{Z})$ is defined as follows: the $\mathbb{Z}$-module $Z_{k-1}(M;\mathbb{Z})$ is free, hence projective, and the map $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ is a surjective $\mathbb{Z}$-module homomorphism. Thus, we have a lifting

$$\begin{array}{ccc} \mathbb{R} & \twoheadrightarrow & \mathbb{R}/\mathbb{Z} \\ & \overline{\chi} \nwarrow & \uparrow \chi \\ & & Z_{k-1}(M;\mathbb{Z}) \end{array}$$

Now define $I(\overline{\chi}) : C_k(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$ by

$$I(\overline{\chi})(c) = \int_c curv(\chi) - \overline{\chi}(\partial c)$$

This map is well-defined since both summands are well-defined. We also know that $d(curv(\chi)) = d\omega = 0$ where $curv(\chi) \in \Omega^k_{\mathbb{Z}}(M)$. Moreover, $\overline{\chi} \circ \partial$ is a integer-valued function, $d(\overline{\chi}(\partial c)) = 0$. Thus, $I(\overline{\chi})$ defines an integer-valued cocycle. Consider another lift $\tilde{\chi}$. Then, $\overline{\chi} \circ \partial$ and $\tilde{\chi} \circ \partial$ define the same integer and so, $I(\overline{\chi})$

and $I(\tilde{\chi})$ describe the same cocycle. Thus, $[I(\overline{\chi})] = [I(\tilde{\chi})] \in H^k(M;\mathbb{Z})$. This is the characteristic class of χ.

Next, recall the universal coefficient theorem

$$0 \longrightarrow Ext^1_{\mathbb{Z}}(H_{k-2}(M;\mathbb{Z}), \mathbb{R}/\mathbb{Z}) \longrightarrow H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \overset{\langle -,- \rangle}{\longrightarrow} Hom_{\mathbb{Z}}(H_{k-1}(M;\mathbb{Z}), \mathbb{R}/\mathbb{Z}) \longrightarrow 0$$

with the map $\langle -,- \rangle$ given by, for $[u] \in H^{k-1}(M;\mathbb{R}/\mathbb{Z})$, $\langle u, [z] \rangle = u(z)$ for $[z] \in H_{k-1}(M;\mathbb{Z})$. This map is an isomorphism: the map is clearly surjective by construction. Moreover, the map is injective because $Ext^1_{\mathbb{Z}}(H_{k-2}(M;\mathbb{Z}), \mathbb{R}/\mathbb{Z}) = 0$ (since $\mathbb{R}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module). Now, apply the contravariant functor $Hom_{\mathbb{Z}}(-, \mathbb{R}/\mathbb{Z})$ to the map $P : Z_{k-1}(M;\mathbb{Z}) \twoheadrightarrow H_{k-1}(M;\mathbb{Z})$ yields the inclusion $Hom_{\mathbb{Z}}(H_{k-1}(M;\mathbb{Z}), \mathbb{R}/\mathbb{Z}) \longrightarrow Hom_{\mathbb{Z}}(Z_{k-1}(M;\mathbb{Z}), \mathbb{R}/\mathbb{Z})$. This map is an inclusion, since $\mathbb{R}/\mathbb{Z}$ is an injective module. Thus, we have

$$H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \overset{\sim}{\longrightarrow} Hom_{\mathbb{Z}}(H_{k-1}(M;\mathbb{Z}), \mathbb{R}/\mathbb{Z}) \hookrightarrow Hom_{\mathbb{Z}}(Z_{k-1}(M;\mathbb{Z}), \mathbb{R}/\mathbb{Z})$$

Now, given $\overline{\chi} \in Hom_{\mathbb{Z}}(H_{k-1}(M;\mathbb{Z}), \mathbb{R}/\mathbb{Z})$, we have a unique $[\omega] \in H^{k-1}(M;\mathbb{R}/\mathbb{Z})$. The correspondence is modulo $\mathbb{Z}$. Moreover, $\chi \in Hom_{\mathbb{Z}}(Z_{k-1}(M;\mathbb{Z}), \mathbb{R}/\mathbb{Z})$ such that $\overline{\chi}(z + \partial b) = \chi(z) \mod \mathbb{Z}$. Thus, $\overline{\chi}$ is a secretly a differential character and hence we have the diagram

$$\begin{array}{ccccc} & & & & \hat{H}^k(M;\mathbb{Z}) \\ & & & \nearrow & \downarrow \\ H^{k-1}(M;\mathbb{R}/\mathbb{Z}) & \longrightarrow & Hom_{\mathbb{Z}}(H_{k-1}(M;\mathbb{Z}), \mathbb{R}/\mathbb{Z}) & \hookrightarrow & Hom_{\mathbb{Z}}(Z_{k-1}(M;\mathbb{Z}), \mathbb{R}/\mathbb{Z}) \end{array}$$

The composite $\langle -,- \rangle : H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \longrightarrow \hat{H}^k(M;\mathbb{Z})$ gives us another map for the digram.

Now, for the final map, consider $\imath : \Omega^{k-1}(M) \longrightarrow Hom_{\mathbb{Z}}(Z_{k-1}(M;\mathbb{Z}), \mathbb{R}/\mathbb{Z})$ by

$$\imath(\omega)(z) := \exp\left(2\pi i \int_z \omega\right)$$

where $i^2 = -1$ and z is a smooth $(k-1)$-cycle. This map is well-defined, since $\imath(\omega)(z) = \cos 2\pi\theta_{\omega,z} + i \sin 2\pi\theta_{\omega,z}$ for where $\theta_{\omega,z}$ depends on ω and z and is given by $\theta_{\omega,z} = \int_z \omega$. Note that if c is a k-chain, then

$$\imath(\omega)(\partial c) = \exp\left(2\pi i \int_{\partial c} \omega\right) = \exp\left(2\pi i \int_c d\omega\right)$$

In fact,

$$\imath(\omega)(z) = \int_z \omega \mod \mathbb{Z} \text{ and } \imath(\omega)(\partial c) = \int_c d\omega \mod \mathbb{Z}$$

Thus, $im(\imath) = \hat{H}^k(M;\mathbb{Z})$ and $curv \circ \imath = d$ and we have commutativity in the diagram. To show that $ch \circ \imath = 0$, note that

$$I(\theta_{\omega,z}) = \int_z curv(\imath\omega) - \theta_{\omega,\partial z} = \int_z d\omega - \int_{\partial z} \omega = \int_z d\omega - \int_z d\omega = 0$$

Next, to find $\ker \imath$, we note the following equivalences

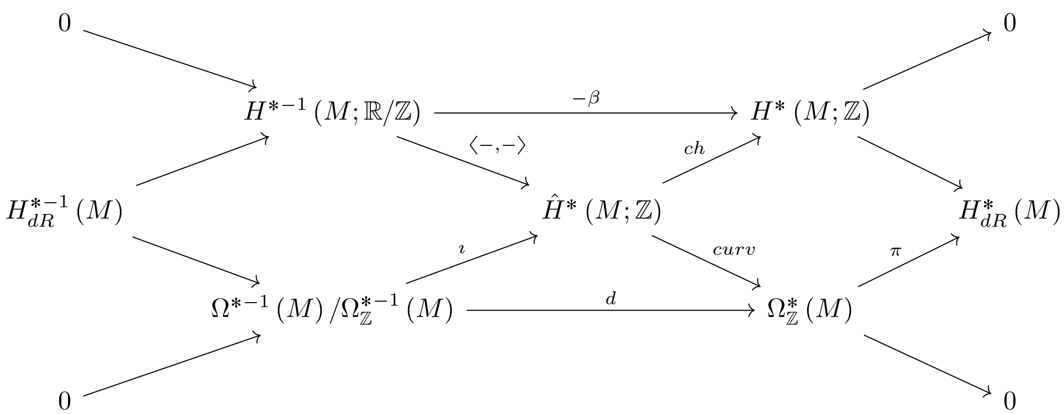
$$\imath(\omega)(z) \quad = \quad 0 \mod \mathbb{Z} \forall z \iff \exp\left(2\pi i \int_z \omega\right) \in \mathbb{Z} \ \forall z$$

$$\iff \quad \int_z \omega \in \mathbb{Z}\ \forall z$$

Moreover, ω is closed since for any cycle z, $\imath(d\omega)(z) = \imath(\omega)(\partial z)$ by Stokes' Theorem. Since z is a cycle, $\partial z = 0$ and

$$\int_0 \omega = 0$$

Thus, $\ker \imath = \Omega^{k-1}_{\mathbb{Z}}(M)$. Thus, the differential cohomology diagram, with exact diagonals, is



(11)

Theorem 4 *Let $\mathcal{M}an$ be the category of smooth manifolds and $\mathcal{G}r\mathcal{A}b$, the category of graded, abelian groups. There is a unique functor[8] $\hat{H}(-;\mathbb{Z}) : \mathcal{M}an^{op} \longrightarrow \mathcal{G}r\mathcal{A}b$ equipped with natural transformations*

1. $\langle -,- \rangle : \hat{H}^{*-1}(-;\mathbb{R}/\mathbb{Z}) \longrightarrow \hat{H}^*(-;\mathbb{Z})$,
2. $\imath : \Omega^{*-1}(-)/\Omega^{*-1}_{\mathbb{Z}}(-) \longrightarrow \hat{H}^*(-;\mathbb{Z})$,
3. $ch : \hat{H}^*(-;\mathbb{Z}) \longrightarrow \hat{H}^*(-;\mathbb{Z})$, *and*
4. $curv : \hat{H}^*(-;\mathbb{Z}) \longrightarrow \Omega^{*-1}_{\mathbb{Z}}(-)$

satisfying diagram 11

Properties of $\hat{H}^\bullet$

In general, since $\hat{H}^k(M)$ is a topological space in particular, the set of path components of $\hat{H}^k(M)$ is well-defined and a fundamental property of Differential Cohomology is that $\pi_0\left(\hat{H}^k(M)\right) = H^k(M;\mathbb{Z})$. Thus, F and F' may be seen as elements in a different path components of $\hat{H}^2(M)$. Moreover, $\hat{H}^k(M)/T^k(M)$ is isomorphic to the collection of exact k-forms $\Omega^k(M)$ on M as a vector space. Here, $T^k(M) := H^{k-1}(M;\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z}) \subset \hat{H}^k(M)$, where $\mathbb{R}/\mathbb{Z} \cong U(1)$, the circle group[1].

Appendix: Some Differential Geometry

Let $E \longrightarrow M$ be a complex vector bundle and $\Omega^k(M,\mathbb{C}) := \Omega^k(M) \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the de Rahm complex $\Omega^k(M)$ and let $\Omega^*(M,E)$ be the graded $\Omega^*(M,\mathbb{C})$-module of differential forms with coefficients in E. A connection on E is a map $\nabla : \Omega^0(M,E) = \Gamma(M,E) \longrightarrow \Omega^1(M,E)$ such that

[1]This gives rise to the name of the group "Torus group" T^q

$\nabla (f \wedge \phi) = df \wedge \phi + f \wedge \nabla\phi \ \forall f \in \Omega^0 (M, \mathbb{C})$ and $\forall \phi \in \Omega^0 (M, E)$. Therefore, a connection on a trivial line bundle is just a 1-form A. The curvature in this case is the 2-form $F = dA$.

Every vector bundle admits a connection. Connections on E uniquely extend to a linear map $\nabla : \Omega^k (M, E) \longrightarrow \Omega^{k+1} (M, E)$ satisfying $\nabla (\omega \wedge \phi) = d\omega \wedge \phi + (-1)^k \omega \wedge \nabla\phi \ \forall \omega \in \Omega^k (M, \mathbb{C})$ and $\forall \phi \in \Omega^k (M, E)$.

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