Problem of the coexistence of several non-Hermitian observables in $\mathcal{PT}$-symmetric quantum mechanics

Miloslav Znojil, Iveta Semorádová, Frantiusek Ruziucka, Hafida Moulla, and Ilhem Leghrib

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Abdullah Naeem Malik

Delaware State University

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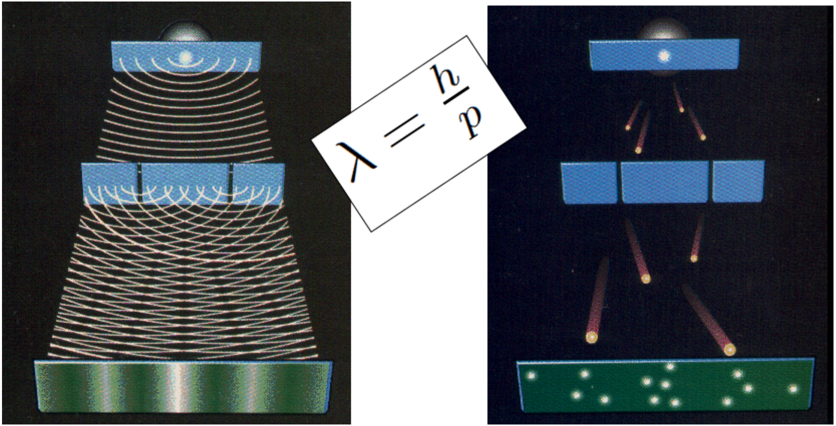
"Everything which is not forbidden is compulsory" Gell-Mann

Overview

- Basic Quantum Mechanics
- Some Terminology
- Background
  - PT-Symmetric Quantum Theory[1]
  - Complex Extension of Quantum Mechanics[2]
- Abstract of the paper
- Introduction of the paper
- Main result of the paper

Physical Motivation for Quantum Mechanics



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- Axiom 4: $\langle A \rangle = \langle x, Ax \rangle = \int \overline{x} Axdt$

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- Can Hamiltonian be $\mathcal{PT}$-Symmetric?[1]

$\mathcal{PT}$-Symmetry and Quantum Mechanics[1]

- Important element: $i\hat{x}$ e.g. $H = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2\left(i\hat{x}\right)^\delta$

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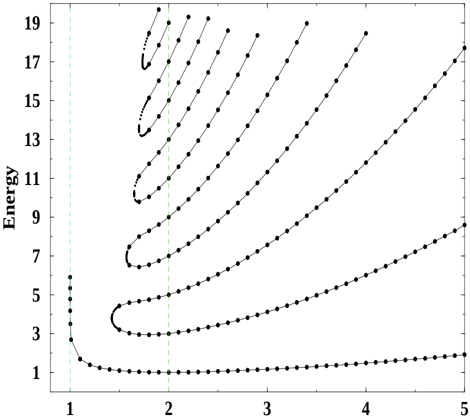
- $\mathcal{PT}$-Symmetry is broken if $\delta < -2$



$$N = \delta + 2$$

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Proof.

$[H, \mathcal{PT}] = 0 \implies \exists\phi_n : H\phi_n = E\phi_n$ and $\mathcal{PT}\phi_n = \lambda\phi_n \implies (\mathcal{PT})^2\,\phi_n = |\lambda|^2\,\phi_n \implies \lambda = e^{i\theta}$. We can choose $\theta = 0$. Now, $H\phi_n = E\phi_n \implies E\phi_n = E^*\phi_n$[3] □

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$\mathcal{PT}\phi_n = \lambda\phi_n \implies \left(\mathcal{PT}\right)^2\phi_n = |\lambda|^2\phi_n \implies \lambda = e^{i\theta}$. We can choose $\theta = 0$.
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- $\Sigma\left(-1\right)^n\phi_n\left(x\right)\phi_n\left(y\right) = \delta\left(x - y\right)$

$\mathcal{PT}$-Symmetry and Quantum Mechanics[1]

- $H = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2\left(i\hat{x}\right)^{\delta}$ has real eigenvalues for all $\delta \geq 0$ (note: H is $\mathcal{PT}$-symmetric but not Hermitian for $\delta \neq 0$)

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- $\Sigma\left(-1\right)^n\phi_n\left(x\right)\phi_n\left(y\right) = \delta\left(x - y\right)$
- Can we now have a new condition: $H = H_{\mathcal{PT}}$ instead of $H = H^*$?

Complex Extension of Quantum Mechanics[2]

- If $\mathcal{PT}$-Symmetry is not broken, then is $\|H_{\mathcal{PT}}f\| = \|f\|$?

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- Choice: $\mathcal{C} = e^{Q(\hat{x}, \hat{p})}\mathcal{P}$ such that $Q(\hat{x}, \hat{p}) = -Q(-\hat{x}, -\hat{p})$ and $[\mathcal{C}, H] = 0$. Then, $\mathcal{C}^2 = 1$, $[\mathcal{C}, \mathcal{P}] \neq 0$ but $[\mathcal{C}, \mathcal{PT}] = 0$ so that $\|\phi\|_{\mathcal{CPT}} = 1$

Abstract

During the recent developments of quantum theory it has been claried that the observable quantities (like energy or position) may be represented by operators Λ (with real spectra) which are manifestly non-Hermitian in a preselected "friendly" Hilbert space $H^{(\mathcal{F})}$. The consistency of these models is known to require an upgrade of the inner product, i.e., mathematically speaking, a transition $H^{(\mathcal{F})} \to H^{(\mathcal{S})}$ to another, "standard" Hilbert space. We prove that whenever we are given more than one candidate for an observable (i.e., say, two operators Λ_0 and Λ_1) in advance, such an upgrade *need not* exist in general.

Introduction

- "There exists a subtle correspondence between the choice of the so called irreducible sets of the candidates Λ_j for the observables and the role played by these sets in the removal of the well known ambiguity of the assignment of the ... Hilbert space $H^{(\mathcal{F})}$ to a single observable Λ_0"

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- When is $\Theta_j = \Theta_i$? Obvious: $[\Theta_j, \Theta_i] = 0$. Another possibility: $\Lambda_j \Theta_j = \Theta_j \Lambda_j^*$

Main Result

- Past approaches: let $\hat{x} \longmapsto \hat{X}$, $\hat{p} \longmapsto \hat{P}$ such that $\hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}) \longmapsto \frac{1}{\sqrt{2}}\left(\hat{X} + i\hat{P}\right)$ and $\hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p}) \longmapsto \frac{1}{\sqrt{2}}\left(\hat{X} - i\hat{P}\right)$ conditional to $\hat{a}\hat{a}^\dagger - q\hat{a}^\dagger\hat{a} = \left[\hat{a}, \hat{a}^\dagger\right]_q = I$. Let $q = 1 - \epsilon$.

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- Then, $\hat{X} = \hat{x} + \frac{1}{8}\epsilon X_1 + \mathcal{O}\left(\epsilon^2\right)$ with $X_1 = \hat{x}^3 - \hat{x}\hat{p}^2 + i\hat{x}^2 + i\hat{x}^2\hat{p} - \hat{x} + i\hat{p}^3 + \hat{p}\hat{x}\hat{p} + \hat{p}^2\hat{x}$ et.c

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- Also, $H_q = \hat{P}^2 + \hat{X}^2 = \hat{p}^2 + \hat{x}^2 + \frac{1}{8}\epsilon H_1$ with $H_1 = 2\hat{x}^4 - \hat{x}^2 + 3\hat{p}^2 - 3 + 2i\hat{x}^3\hat{p} + 2i\hat{x}\hat{p}^3 + 2\hat{x}^2\hat{p}^2 - 8i\hat{x}\hat{p}$.

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- $H_1 \neq H_1^*$ and $X_1 \neq X_1^*$

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