MAC1140 Pre-Calculus Algebra

Contents

TOPIC 0

Review

0.1. Complex Numbers (ALEKS R.6)

One property of any given real number is that its square is always nonnegative. That is, there is no real number x such that $x^2 = -1$. To remedy this situation, we introduce a new number called an imaginary unit.

Definition 0.1.1: Imaginary Unit

The **imaginary unit**, i, is defined to be a solution to the equation $x^2 = -1$. The solution set for the equation $x^2 = -1$ is $\{i, -i\}$, where i is the **principal square root** of -1.

Thus, $i^2 = -1$ by definition, and we use the notation $i = \sqrt{-1}$.

Objective 1: Simplifying imaginary numbers

The powers of i repeat with every fourth power with even powers of i resulting in 1 or -1, whereas odd powers of powers of i result in i or $-i$.

Example 0.2. *Simplify* $\sqrt{-12}$

Example 0.3. *Simplify the given powers of* i*.*

A $i^3 =$

B $i^4 =$

C $i^5 =$

D $i^{-30} =$

E $i^{64} =$

F $i^{103} =$

Objective 2: Operations with complex numbers

Addition, subtraction, multiplication and division of complex numbers follows a natural set of rules, as discussed below.

Definition 0.3.1: Complex Number

A **complex number** is a number that may be written in the form $a + bi$ where a and b are real numbers. This form is called the **standard form** of a complex number. The number a is called the ________________ part and bi is called the ________________ part, where a and b are real numbers.

Example 0.4. *Write the following numbers in standard $a + bi$ form.*

A $4i - (3 + 2i) + (1 - 3i)$

B $(4 + 2i)(3 - i)$

C $(3 + 4i)^2$

Definition 0.4.1: Complex Conjugate

The **complex conjugate** of $z = a + bi$ is $\overline{z} =$ ______________________________

Remark. *Notice that the product of conjugates is a real number. In particular,*

$z\overline{z} = (a + bi)(a - bi) =$ ____

Example 0.5. *Write the following numbers in $a + bi$ form.*

A $\dfrac{4 + 2i}{3 - i}$

B $\dfrac{4}{3 + 5i}$

Objective 3: Solve quadratic equations over the set of complex numbers

The two (possibly repeated) solutions, to the equation $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

EXAMPLE 0.6. *Find all complex zeros of* $f(x) = x^4 + 27x$.

EXAMPLE 0.7. *Solve* $x^4 - 5x^2 + 6 = 0$.

Objective 4: Using the discriminant to determine nature of solutions

The discriminant is a number associated with quadratic equations, which tells us the number and type of solutions to the equation.

Definition 0.7.1: Discriminant

The **discriminant** of the quadratic equation $ax^2 + bx + c = 0$ is ________________
If the **discriminant** of a quadratic equation is

- **A** zero, there is ________________________________
- **B** positive, there are ______________________________
- **C** negative, there are _____________________________

EXAMPLE 0.8. *Find the discriminant and give the number and type of solution(s) of the quadratic equation.* $2x^2 - 3x + 7 = 0$

TOPIC I

Polynomial and Rational Functions

I.1. Quadratic Functions (ALEKS 2.1)

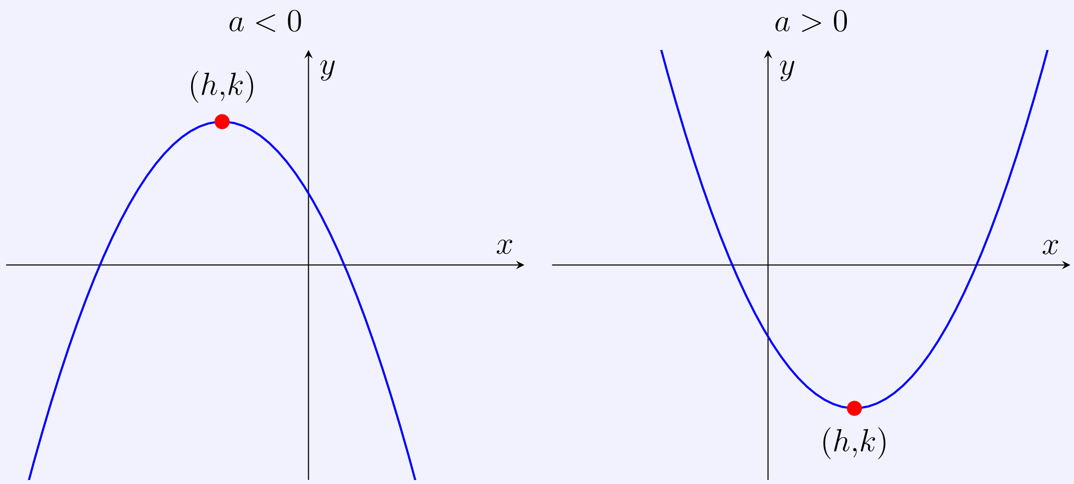
Objective 5: Graph a quadratic function written in vertex form

The graph of the quadratic function $f(x) = ax^2 + bx + c$, for $a \neq 0$, is a ____________.

The parabola opens upward for ______________ and opens downward for ______________.

For a function $f(x) = a(x-h)^2 + k$ with $a \neq 0$, the vertex is _______________________.

Consider the functions, with given values of a



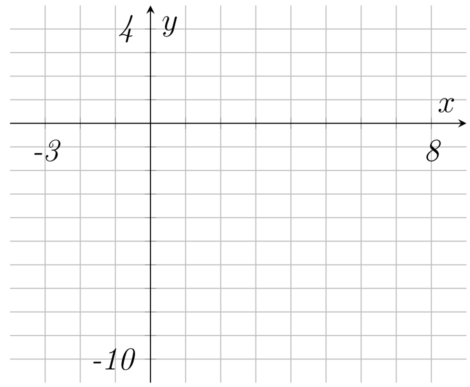
The axis of symmetry is ____

The domain of the function is ____

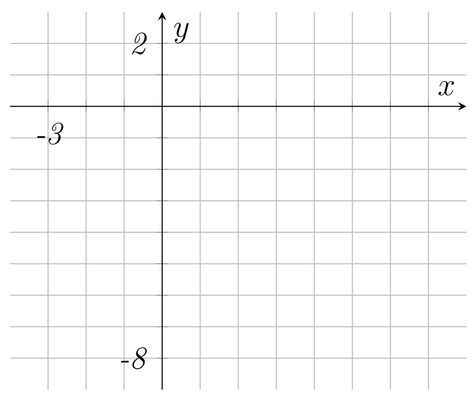
If $a < 0$, the maximum value of the function is k and the range is ____________.

If $a > 0$, the minimum value of the function is k and the range is ____________.

Example I.2. *Graph $f(x) = (x-3)^2 - 9$. Identify the vertex, x- and y-intercepts, and axis of symmetry.*



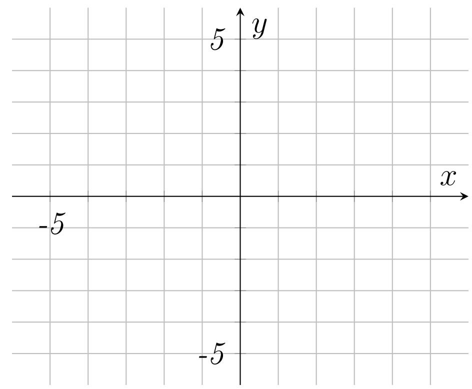
Example I.3. *Graph $f(x) = -(x-2)^2 - 1$. Identify the vertex, x- and y-intercepts, and axis of symmetry.*



Objective 6: Write quadratic functions in vertex form

You will use the technique of completing the square to write $Ax^2 + Bx + C$ as $a(x-h)^2 + k$, where a, h and k and constants determined by A, B and C.h

Example I.4. *Write $f(x) = -x^2 - 2x + 3$ in vertex form and graph the function. Identify the vertex, x- and y-intercepts, and axis of symmetry. Write the domain and range in interval notation.*



EXAMPLE I.5. *For* $f(x) = ax^2 + bx + c$ *(where* $a \neq 0$*),* $b^2 - 4ac$ *is called* ________.

A *If* $b^2 - 4ac = 0$*, the graph of* $y = f(x)$ *has* ________________________________.

B *If* $b^2 - 4ac > 0$*, the graph of* $y = f(x)$ *has* ________________________________.

C *If* $b^2 - 4ac < 0$*, the graph of* $y = f(x)$ *has* ________________________________.

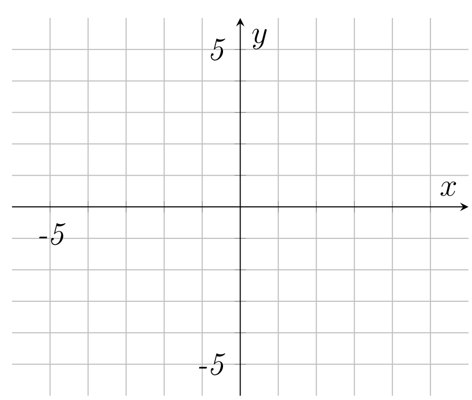
Objective 7: Find the vertex of a parabola by using the vertex formula

For $f(x) = ax^2 + bx + c$ (where $a \neq 0$), the x-coordinate of the vertex is given by $x = -\frac{b}{2a}$.

To find the y-coordinate, evaluate $f\left(-\frac{b}{2a}\right)$.

The vertex is given by (h, k), where $h =$ ____________ and $k =$ ____________

EXAMPLE I.6. *Given* $f(x) = -3x^2 + 48x - 165$*, determine the vertex by using the vertex formula* ***and*** *by writing in vertex form.*



Objective 8: Solve applications involving quadratic functions

Functions can model real-life phenomenon.

Example I.7. *A quarterback throws a football with an initial velocity of 72 ft/s at an angle of 25°. The height of the ball can be modeled by*

$$h(t) = -16t^2 + 30.4t + 5$$

where $h(t)$ is the height (in feet) and t is the time in seconds after release.

A *Determine the time at which the ball will be at its maximum height.*

B *Determine the maximum height of the ball.*

C *Determine the amount of time required for the ball to reach the receiver's hands, if the receiver catches the ball at a point 3 feet off the ground.*

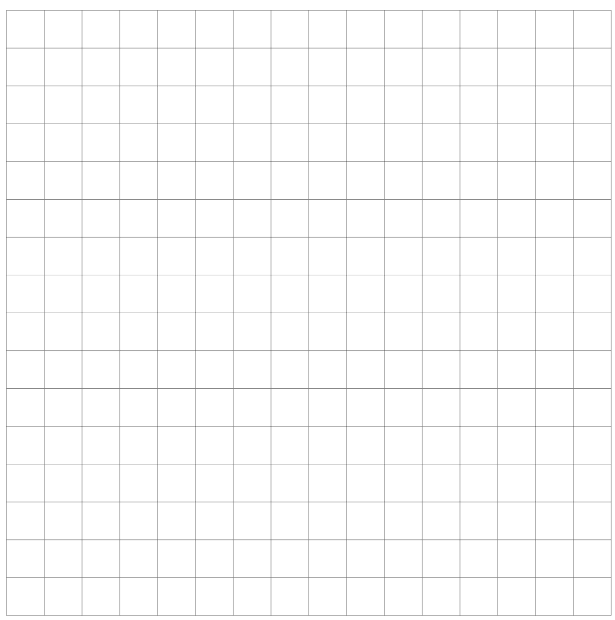
Students - Check your understanding.

CYU 1. *Consider the equation*

$$y = 2x^2 + 20x + 47$$

A *Rewrite the equation in vertex form* $y = a\left(x-h\right)^2 + k$.

B *Graph the quadratic function by finding its vertex, axis of symmetry, y-intercept and x-intercept, if any.*



C *Determine the domain and range.*

D *Determine where the function is increasing and where it is decreasing and where it is decreasing.*

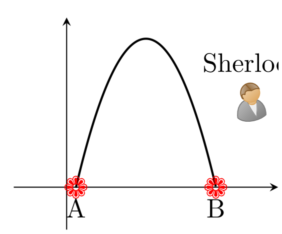
E *Determine whether the given quadratic function has a maximum value or a minimum value and then find the value.*

CYU 2. *At the end of his career, Sherlock Holmes retired to a small farm in the country and took up a hobby of beekeeping. Being of an analytical mind, he could not just watch a bee fly from flower to flower, and so, formulated a mathematical model for the parabolic path the bee follows from flower A to flower B. The path is given by*

$$f(x) = -0.03x^2 + 1.8x - 24$$

where x is the distance in inches along the ground from Sherlock

A What is the maximum height the bee reaches as it flies from A to B? How far away from Sherlock are flowers A and B?



B What is the maximum height the bee reaches as it flies from A to B? How far away from Sherlock are flowers A and B?

I.8. Polynomial Functions(ALEKS 2.2)

Definition I.8.1: Polynomial function

A **polynomial function of degree** n is a function that can be expressed as

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

where n is a non-negative integer and $a_n \neq 0$

The given expression is called **standard form**, __ is the **leading term**, and __ is the **constant term**.

The **domain** of a polynomial is ____________________________________.

The **graph** of a polynomial is ____________________________________.

Objective 9: Determine the end behavior of a polynomial function

To analyze polynomial functions with multiple terms, one important characteristic is the "end behavior" of the function. This involves finding the general direction a function $f(x)$ follows as x approaches ∞ or $-\infty$.

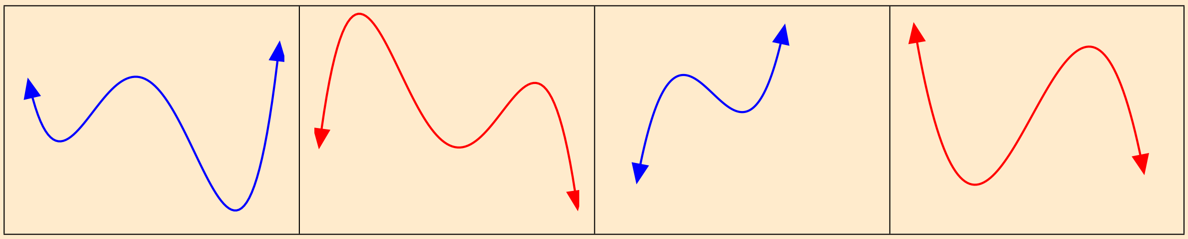
Notation for end behavior

The symbol..	is read as..	which means that..
$x \to \infty$	"x approaches infinity"	x becomes infinitely "large" in the positive direction
$x \to -\infty$	"x approaches negative infinity"	x becomes infinitely "large" in the negative direction
$f(x) \to \infty$	"$f(x)$ approaches infinity"	the y value becomes infinitely "large" in the positive direction
$f(x) \to -\infty$	"$f(x)$ approaches negative infinity"	the y value becomes infinitely "large" in the negative direction

For a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, we can observe the general direction the function follows by looking at the leading term $a_n x^n$. This is because, in some sense, this leading term dominates the behavior of the remaining terms.

The Leading Term Test

$a_n x^{\texttt{even}}$		$a_n x^{\texttt{odd}}$	
$a_n > 0$	$a_n < 0$	$a_n > 0$	$a_n < 0$
As $x \to \infty$,	As $x \to \infty$,	As $x \to \infty$,	As $x \to \infty$,
$f(x) \to$__________	$f(x) \to$__________	$f(x) \to$__________	$f(x) \to$__________
As $x \to -\infty$,	As $x \to -\infty$,	As $x \to -\infty$,	As $x \to -\infty$,
$f(x) \to$__________	$f(x) \to$__________	$f(x) \to$__________	$f(x) \to$__________



Example I.9. *Use the leading term to determine the end behavior of the graph of the function*

A $f(x) = x^5 - 5x^3 + 4x$

B $g(x) = -2(x-5)(x+1)^3$

Objective 10: Identify zeros and multiplicities of zeros

The values of x in the domain of f for which $f(x) = 0$ are called the zeros of the function. These are the real solutions (or roots) of the equation $f(x) = 0$ and correspond to the x-intercepts of the graph of $y = f(x)$.

Example I.10. *Determine the zeros of the function defined by* $f(x) = x^3 - 5x^2 - 16x + 80$.

Definition I.10.1: Zero of Multiplicity k

If a polynomial function has a factor $(x - c)$ that appears exactly k times, then c is a **zero of multiplicity** k.

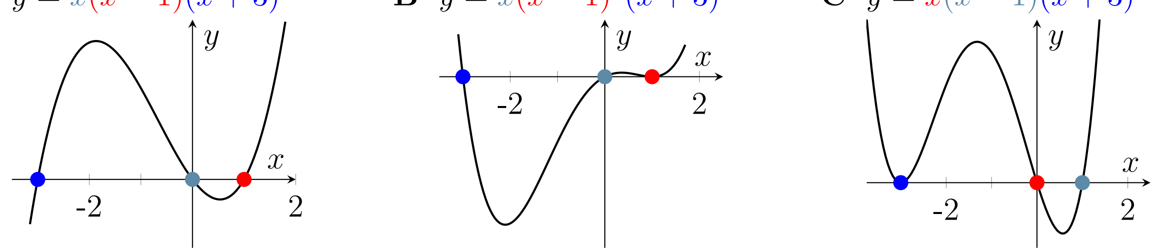
If c is a zero of odd multiplicity, then the graph ____________________ the x-axis at c.

If c is a zero of even multiplicity, then the graph ____________________ the x-axis at c.

Example I.11. *Determine the zeros and their multiplicities for* $f(x) = 8x(x-1)^2(x+3)^3$.

Example I.12. *What effect do the multiplicities of the zeros have on the graph?*

A $y = x(x-1)(x+3)$ **B** $y = x(x-1)^2(x+3)$ **C** $y = x(x-1)(x+3)^2$

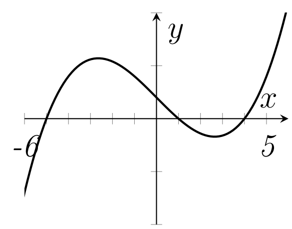


Example I.13. *Given* $f(x) = -3x^2(x+4)(x-7)^3$*, identify the zeros of the function and whether there is a cross point or a touch point at each zero.*

Objective 11: Identifying turning points

The turning points of a polynomial function are the points where the function changes from increasing to decreasing or vice versa. The ____________________ correspond to points where the function has relative maxima and relative minima. The graph of a polynomial function of degree n has at most ____________________ turning points.

EXAMPLE I.14. *Give the maximum number of turning points for* $f(x) = x^3 - 21x + 20$



Objective 12: Sketch a Polynomial Function

To graph a polynomial function defined by $y = f(x)$,

(1) Use the leading term to determine the end behavior of the graph.

(2) Determine the y-intercept by evaluating $f(0)$.

(3) Determine the real zeros of f and their multiplicities (these are the x-intercepts of the graph of f).

(4) Plot the x- and y-intercepts and sketch the end behavior.

(5) Draw a sketch starting from the left-end behavior. Connect the x- and y-intercepts in the order that they appear from left to right. Remember: cross the x-axis if the zero has an odd multiplicity, touch but not cross if the zero has an even multiplicity.

(6) If the test for symmetry is easy to apply, plot additional points.

- f is an even function (symmetric to the y-axis) if $f(-x) = f(x)$.

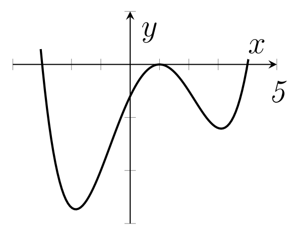
- ∘ f is an odd function (symmetric to the origin) if $f(-x) = -f(x)$.

(7) Plot more points if a greater level of accuracy is desired. In particular, to estimate the location of Turning points, find several points between two consecutive x-intercepts.

EXAMPLE I.15. *Sketch the function* $f(x) = (x-1)^2(x+3)(x-4)$

General shape? *Turns?*

Intercepts? *Cross or touch?*



Objective 13: Applying the Intermediate Value Theorem

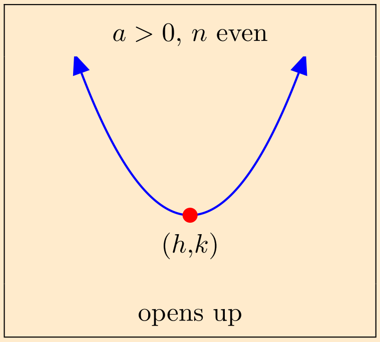
The Intermediate Value Theorem can be used to find zeros of polynomial functions.

Theorem I.15.1: Intermediate Value Theorem

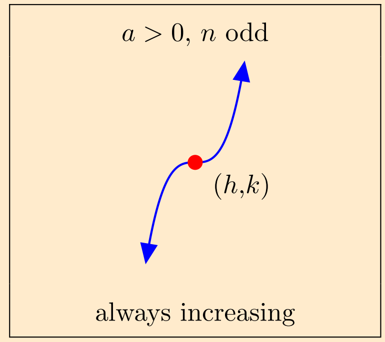
Let f be a polynomial function. For $a < b$, if $f(a)$ and $f(b)$ have opposite signs, then f has at least one zero in the interval (a, b).

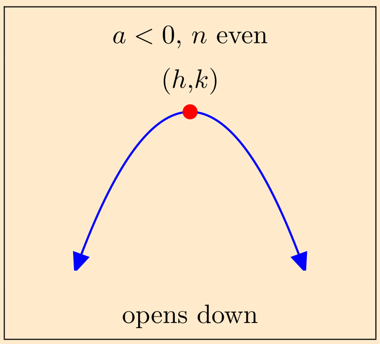
EXAMPLE I.16. *Show that* $f(x) = x^3 - 13x^2 + 33x + 35$ *has a zero in the interval* $(-1, 0)$.

Transformations of $y = x^n$ as $y = a\left(x-h\right)^n + k$

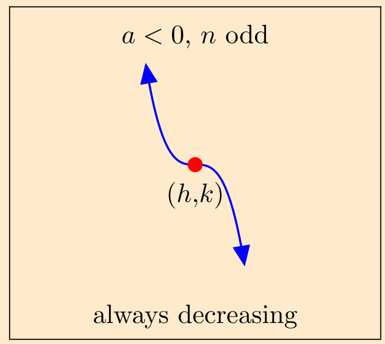


- $a < 0$; Reflect over x-axis
- a scales vertically



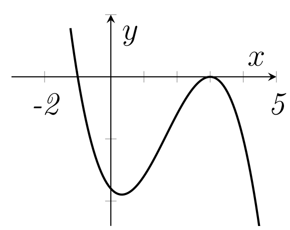


- h shifts horizontally
- k shifts vertically



Students - Check your understanding.

CYU 3. *Consider the function* $f\left(x\right) = -\left(x+1\right)\left(x-3\right)^2$. *Follow through the steps of sketching a graph. Your steps should match the graph on the right.*



I.17. Division of Polynomials (ALEKS 2.3)

Much like division with numbers, involving dividends, divisors, quotients and remainders, there is a complete analogy involving polynomials.

Objective 14: Divide polynomials using long division

Suppose that $f(x)$ and $d(x)$ are polynomials where $d(x) \neq 0$ and the degree of $d(x)$ is less than or equal to the degree of $f(x)$. Then there exists unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)\,q(x) + r(x) \;\text{ or }\; \frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

where either the degree of $r(x)$ is less than $d(x)$, or $r(x)$ is the zero polynomial.

Definition I.17.1: Polynomial Division

The polynomial $f(x)$ is the **dividend**, $d(x)$ is the **divisor**, $q(x)$ is the **quotient**, and $r(x)$ is the **remainder**.

To begin the division: always write the terms of the dividend and divisor in descending order with zeros for place holders for missing powers of x.

Example I.18. *Use long division to divide*

A $(2x^3 - 11x^2 + 13x + 15) \div (2x - 5)$ **B** $(x^5 - 3x^4 + 12x - 33) \div (x - 3)$

Objective 15: Divide polynomials using synthetic division

When dividing polynomials where the divisor is a binomial of the form $(x - c)$ and c is a constant, we can use synthetic division.

EXAMPLE I.19. $(x^3 - 13x^2 + 46x - 49) \div (x - 8)$

The setup:

Step 1: Write the value of c in a box.

Step 2: Write the coefficients of the dividend to the right of the box.

Step 3: Skip a line and draw a horizontal line below the list of coefficients

Step 4: Bring down the leading coefficient from the dividend and write it below the line.

The division:

Step 5: Multiply the value of c by the number below the line. Write the result in the next column above the line.

Step 6: Add the numbers in the column above the line, and write the result below the line.

Repeat steps 5 and 6 until all columns have been completed.

The right-most number below the line is the remainder. The other numbers below the line are the coefficients of the quotient in order by the degree of the term. Since the divisor is linear (1^{st} degree), then the degree of the quotient is 1 less than the degree of the dividend.

Objective 16: Apply the Remainder and Factor Theorems

The value of $f(c)$ is the same as the remainder we get from dividing $f(x)$ by $x - c$. On the other hand, the factor theorem tells us when $x - c$ is a factor of $f(x)$

Theorem I.19.1: Remainder Theorem

If a polynomial $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

EXAMPLE I.20. *Given* $f(x) = 5x^7 - 3x^5 + 2x^2 - 1$*, use the remainder theorem to evaluate* $f(-1)$*.*

EXAMPLE I.21. *Use the remainder theorem to show that* $\sqrt{5}$ *is a zero of* $f(x) = x^3 - 4x^2 - 5x + 20$*.*

Polynomials may also be evaluated over the set of complex numbers rather than restricting x to the set of real numbers. A complex number a + bi is a zero of a polynomial $f(x)$ **if** $f(a+bi) = \mathbf{0}$**.**

EXAMPLE I.22. *Use the remainder theorem to show that* $8i$ *is a zero of* $f(x) = 2x^3 + x^2 + 128x + 64$*.*

Theorem I.22.1: Factor Theorem

Let $f(x)$ be a polynomial. If $f(c) = 0$, then $(x - c)$ is a factor of $f(x)$. If $(x - c)$ is a factor of $f(x)$, then $f(c) = 0$..

CYU 4. *Use the factor theorem to determine if* $(x + 4)$ *is a factor of* $f(x) = 6x^3 + 11x^2 - 60x - 32$

EXAMPLE I.23. *Factor* $f(x) = 4x^5 - 24x^4 - 3x^3 + 18x^2 - x + 6$, *given that* $(x - 6)$ *is a factor.*

EXAMPLE I.24. *Write a polynomial of degree* 3 *that has the zeros* 2 *(of multiplicity* 2*) and* -5

Students - Check your understanding.

CYU 5. *Use long division to divide* $(2x^3 + 2x^2 - 16x + 32) \div (x^2 - 3x + 4)$

CYU 6. *Divide using synthetic division.*

A $(x^5 - 3x^4 + 12x - 33) \div (x - 3)$

B $(3x^3 - 48x + 25x^2 + 20) \div (x + 10)$

I.25. Zeros of Polynomials (ALEKS 2.4)

Objective 17: Apply the Rational Zero Theorem

Determine a list of rational numbers which may (or may not be) zeroes of a given function

Theorem I.25.1: Rational Zero Theorem

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0$ has integer coefficients and $a_n \neq 0$, and if $\frac{p}{q}$ (written in lowest terms) is a rational zero of f, then p is a factor of and q is a factor of the ______________________________.

If a rational zero exists for a polynomial, then it must be of the form

$$\frac{p}{q} = \frac{\text{Factors of } a_0 \text{ (constant term)}}{\text{Factors of } a_n \text{ (leading coefficient)}}$$

Example I.26. *List all possible rational zeros of* $f(x) = 4x^3 - 13x^2 - 32x - 15$

Example I.27. *Find the zeros and their multiplicities of* $f(x) = 2x^4 + x^3 - 5x^2 - x + 3$

Example I.28. *Find the zeros and their multiplicities* $f(x) = x^3 - 2x^2 - 14x - 5$

Objective 18: Apply the Fundamental Theorem of Algebra

The theorem provides a guarantee that a zero of polynomial will exist. There are four different versions of the theorem, listed below.

Theorem I.28.1: Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree $n \geq 1$ with *complex* coefficients, then $f(x)$ has ___.

Theorem I.28.2: Linear Factorization Theorem

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0$ where $n \geq 1$ and $a_n \neq 0$, then $f(x) = a_n(x - c_1)(x - c_2)...(x - c_n)$ where $c_1, c_2, ..., c_n$ are complex numbers.

Theorem I.28.3: Conjugate Zeros Theorem

If $f(x)$ is a polynomial with *real* coefficients and if $a + bi$ $(b \neq 0)$ is a zero of $f(x)$ then its conjugate ________________________ is also a zero of $f(x)$

Theorem I.28.4: Number of Zeros of a Polynomial

If $f(x)$ is a polynomial of degree $n \geq 1$ with complex coefficients, then $f(x)$ has exactly n complex zeros provided that each zero is counted by its multiplicity.

Example I.29. *Given that $1 - 5i$ is a zero of $f(x) = x^3 - 3x^2 + 28x - 26$, find the remaining zeros and factor $f(x)$ as a product of linear factors.*

Example I.30. *Given that $x = -\frac{1}{2}$ is one solution to $2x^4 - x^3 + 17x^2 - 9x - 9 = 0$, find the remaining solutions.*

Example I.31. *Find a polynomial* $f(x)$ *of lowest degree with zeros of* $3i$ *and* 2 *(multiplicity 2).*

Example I.32. *Find a polynomial* $p(x)$ *of degree* 3 *with zeros* $-2i$ *and* 4*. The polynomial must also satisfy the condition that* $p(0) = 32$*.*

I.33. Rational Functions (ALEKS 2.5)

Definition I.33.1: Rational function

Let $p(x)$ and $q(x)$ be polynomials where $q(x) \neq 0$. A function $f(x)$ defined by

$$f(x) = \frac{p(x)}{q(x)}$$

is called a **rational function**.

Objective 19: Apply Notation Describing Infinite Behavior of a Function

Sometimes, a function $f(x)$ may approach $\pm\infty$, close to a particular value of x; or as x gets larger and larger; or as x gets smaller and smaller.

Notation for infinite behavior

The symbol..	is read as..	which means that..
$x \to c^+$	"x approaches c from the right"	x decreases to values closer to c, but never equals c
$x \to c^-$	"x approaches c from the left"	x increases to values closer to c, but never equals c
$x \to \infty$	"x approaches infinity"	x increases without bound
$x \to -\infty$	"x approaches negative infinity"	x decreases without bound

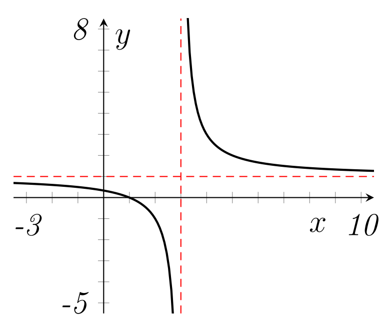
Example I.34. *The graph $f(x) = \frac{x-1}{x-3}$ of is given. Complete the statements.*

A *As* $x \to \infty$, $f(x) \to$ ______________

B *As* $x \to -\infty$, $f(x) \to$ ______________

C *As* $x \to 3^+$, $f(x) \to$ ______________

D *As* $x \to 3^-$, $f(x) \to$ ______________

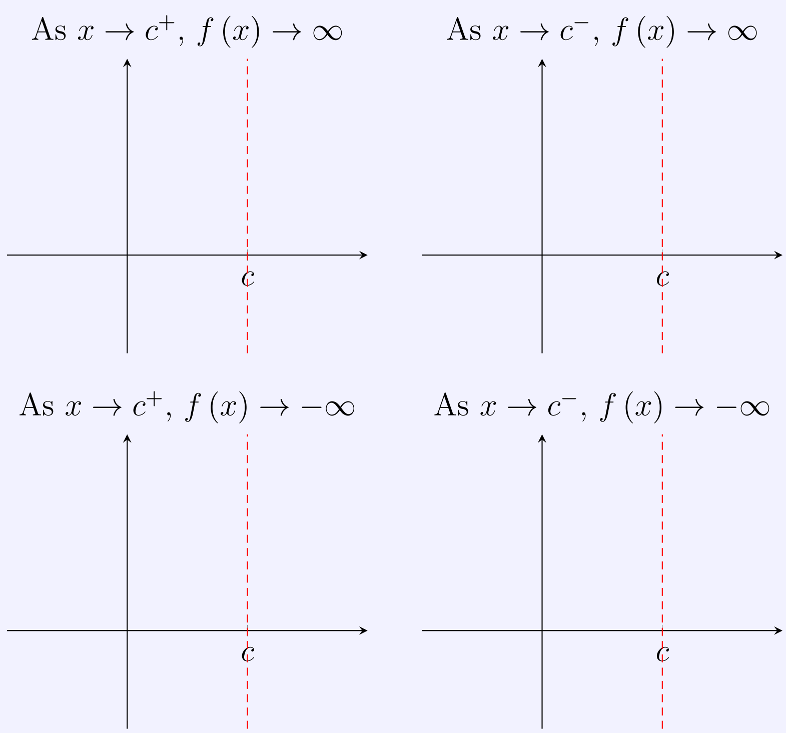


Definition I.34.1: Vertical Asymptote

The line $x = c$ is a **vertical asymptote** of the graph of a function f if $f(x)$ approaches infinity or negative infinity as x approaches c from either side.

Objective 20: Identify vertical asymptotes

Let $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ have no common factors other than 1. To locate the vertical asymptotes of $f(x)$, determine the real numbers x where the denominator is zero, but the numerator is nonzero.



Example I.35. *Identify the vertical asymptotes, if any.*

A $f(x) = \frac{x-3}{x^2-4x-5}$

B $f(x) = \frac{x-3}{x^2-2x-3}$

C $f(x) = -\frac{8}{x^2+25}$

Definition I.35.1: Horizontal Asymptote

The line $y = d$ is a **horizontal asymptote** of the graph of a function f if $f(x)$ approaches d as x approaches infinity or negative infinity.

Objective 21: Identify horizontal asymptotes

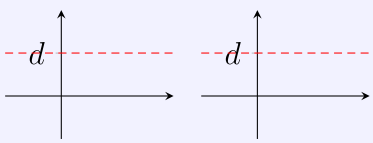
Let f be a rational function defined by $f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0}{b_m x^m + b_{m=1}x^{m-1} + b_{m-2}x^{m-2} + \ldots + b_1 x + b_0}$ where n is the degree of the numerator and m is the degree of the denominator.

(1) If $n > m$, f has ______________________________

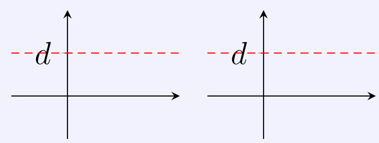
(2) If $n < m$, then the horizontal asymptote of f is ______________

(3) If $n = m$, then the horizontal asymptote of f is ______________

As $x \to \infty$, $f(x) \to d$



As $x \to -\infty$, $f(x) \to d$

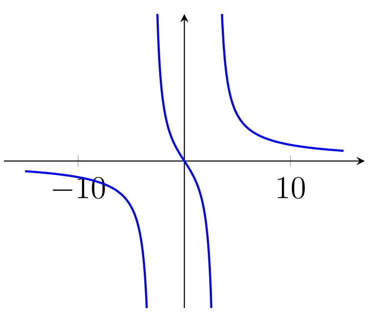


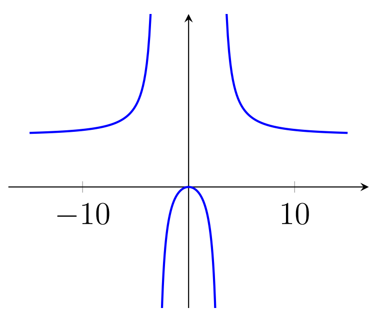
Example I.36. *Identify the horizontal asymptotes, if any.*

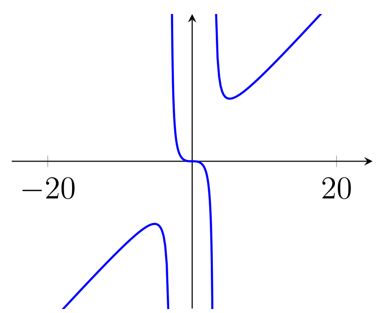
A $f(x) = \frac{3x}{x^2-9}$

B $f(x) = \frac{3x^2}{x-9}$

C $f(x) = \frac{3x^3}{x^2-9}$







Remark. *The graph of a rational function may not cross a vertical asymptote. The graph may cross a horizontal asymptote.*

Example I.37. *Determine if the function* $f(x) = \frac{2x^2-4}{x^2-3x-4}$ *crosses its horizontal asymptote.*

Definition I.37.1: Slant Asymptote

A rational function will have a **slant asymptote** (also known as oblique) if the degree of the numerator is exactly one greater than the degree of the denominator.

Objective 22: Identify Slant Asymptotes

To find an equation of a slant asymptote, divide the numerator of the function by the denominator. The quotient will be linear and the slant asymptote will be of the form $y =$ quotient.

Example I.38. *Determine the slant asymptote of* $f(x) = \frac{3x^3}{x^2-9}$.

Objective 23: Graph Rational Functions

Given a function $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials with no common factors, use the following steps to graph $f(x)$

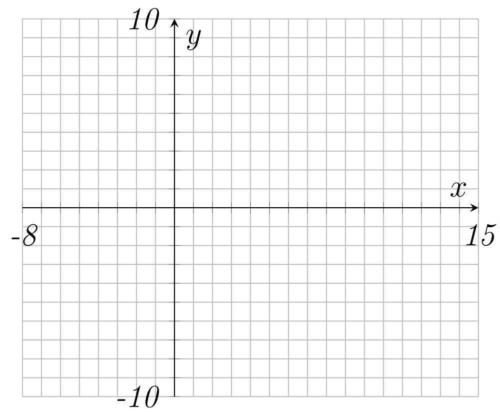
Step 1: Intercepts: Determine the x- and y-intercepts

Step 2: Asymptotes: Determine if the function has vertical asymptotes and graph them as dashed lines.
Determine if the function has a horizontal asymptote or a slant asymptote, and graph the asymptote as a dashed line.
Determine if the function crosses the horizontal or slant asymptote.

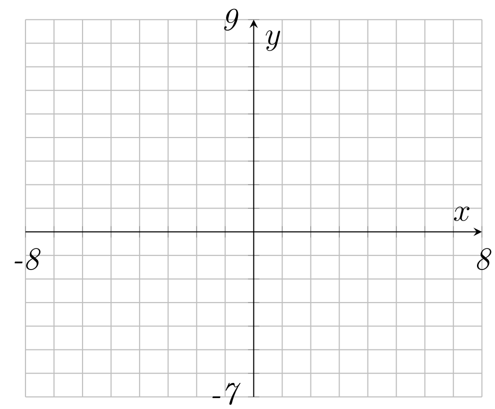
Step 3: Symmetry: f is symmetric to the y-axis if $f(-x) = f(x)$
f is symmetric to the origin if $f(-x) = -f(x)$.

Step 4: Plot points: Plot at least one point on the intervals defined by the x-intercepts, vertical asymptotes, and points where the function crosses a horizontal or slant asymptote.

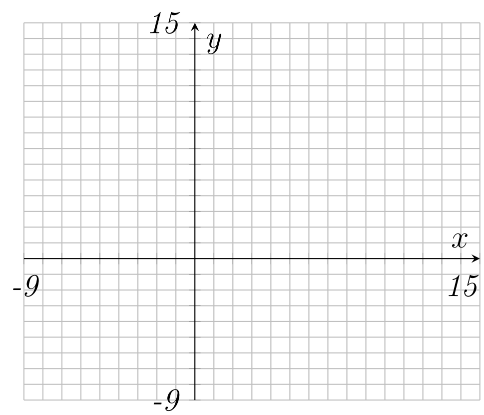
Example I.39. *Graph* $f(x) = \frac{x^2-x-5}{x^2-5x-6}$. *What is the domain and range?*



Example I.40. *Graph* $f(x) = \frac{2x^2}{x^2-4}$. *What is the domain and range?*

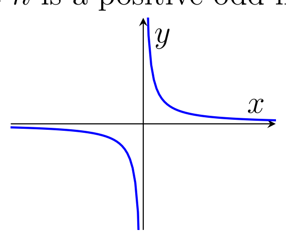


Example I.41. *Graph* $f(x) = \frac{2x^2-8x+8}{x-3}$. *What is the domain and range?*



Basic Rational Functions. The graph of $y = 1/x^n$, where n is a positive odd integer.

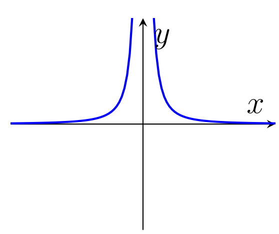
Domain? Asymptotes?



Range? Intercepts?

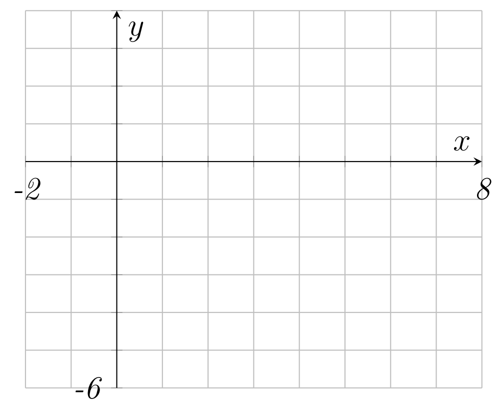
The graph of $y = 1/x^n$, where n is a positive even integer.

Domain? Asymptotes?



Range? Intercepts?

EXAMPLE I.42. *Use transformations to graph* $f(x) = \frac{1}{x-3} - 1$.



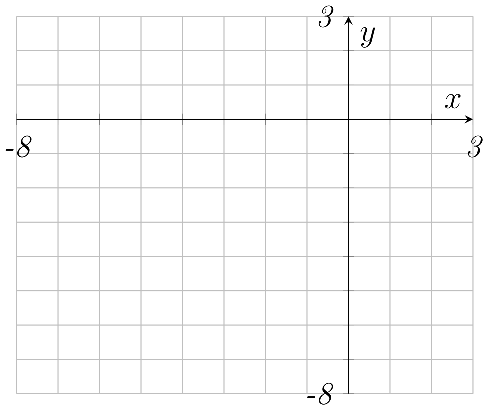
Students - Check your understanding.

CYU 7. *For the rational function* $H(x) = -\frac{1}{x^2+6x+9}$, *find*

A *Domain and range*

B *Vertical asymptote(s), horizontal asymptote(s) and oblique asymptote(s), if any.*

C *Sketch a graph of* $H(x)$



3
y
x
-8
3
-8

I.43. Polynomial and Rational Inequalities (ALEKS 2.6)

Objective 24: Solve Polynomial inequalities

Step 1. Write the inequality so that a polynomial or rational expression f is on the left side and zero is on the right side in one of the following forms:

$$f(x) > 0 \quad f(x) \geq 0$$
$$f(x) < 0 \quad f(x) \leq 0$$

For rational expressions, be sure that the left side is written as a single quotient, and find the domain of f.

Step 2. Determine the real numbers at which the expression f equals zero and, if the expression is rational, the real numbers at which the expression f is undefined.

Step 3. Use the numbers found in Step 2 to separate the real number line into intervals. These numbers are called boundary or partition points.

Step 4. Select a number in each interval and evaluate f at the number.

(a) If the value of f is positive, then $f(x) > 0$ for all numbers x in the interval.

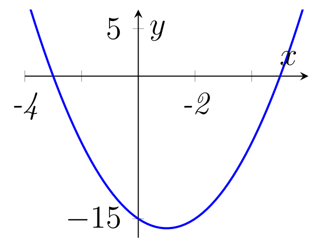
(b) If the value of f is negative, then $f(x) < 0$ for all numbers x in the interval.

If the inequality is not strict ($\leq$ or $\geq$), include the solutions of $f(x) = 0$ that are in the domain of f in the solution set. Be careful to exclude values of x where f is undefined.

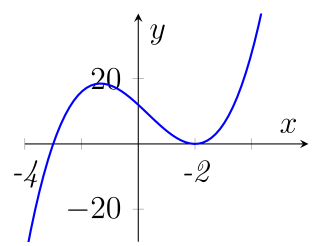
Remark. *This method works because a polynomial or rational functional can only change signs where it equals zero or is undefined.*

Remark. *If a boundary point comes from a linear factor with odd multiplicity, $f(x)$ will change sign on either side of the partition point. If the linear factor has even multiplicity, the sign of $f(x)$ will not change.*

Example I.44. *Solve for x in the inequality $x(x-2) \leq 15$*



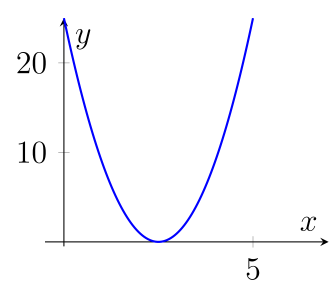
Example I.45. *Solve for x in the inequality $(x-2)^2(x+3) > 0$*



Example I.46. *Solve for x*

A $4x^2 - 20x + 25 \leq 0$

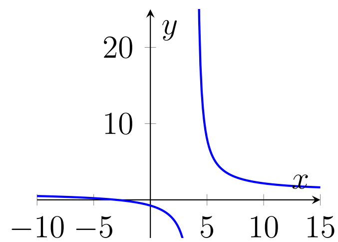
B $4x^2 - 20x + 25 < 0$



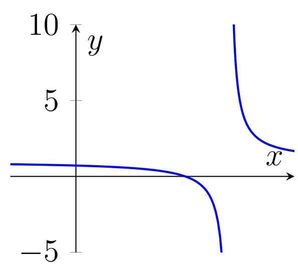
C $4x^2 - 20x + 25 \geq 0$

D $4x^2 - 20x + 25 > 0$

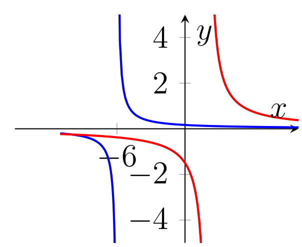
EXAMPLE I.47. *Solve for x in the inequality* $\frac{(x-1)(x+3)}{x-4} \leq 0$



EXAMPLE I.48. *Solve for x in the inequality* $\frac{x-5}{x-7} < 3$



EXAMPLE I.49. *Solve for x in the inequality* $\frac{1}{x+6} \geq \frac{3}{x-2}$



Students - Check your understanding. We can use a combination of the sign chart and information about the graphs of functions that we have learned this semester to make the solution quicker to find.

CYU 8. *Solve* $-2(x+4)^2(4-x)^3 > 0$*. Consider zeros and their multiplicity along with end behavior for the function* $f(x) = -2(x+4)^2(4-x)^3$

CYU 9. *Which of the following intervals satisfy the equation* $\frac{4x-1}{(x+2)(x+3)} \leq 0$

A. $[-3,-2] \cup [1/4, \infty)$

B. $(-3,-2) \cup [1/4, \infty)$

C. $(-\infty,-3] \cup [-2, 1/4]$

D. $(-\infty,-3) \cup (-2, 1/4]$

TOPIC II

Exponential and Logarithmic Functions

II.1. Inverse Functions

Definition II.1.1: One-to-one function

A function f is **one-to-one** if, for a and b in the domain of f, if $a \neq b$ then $f(a) \neq f(b)$. Equivalently, if $f(a) = f(b)$, then $a = b$

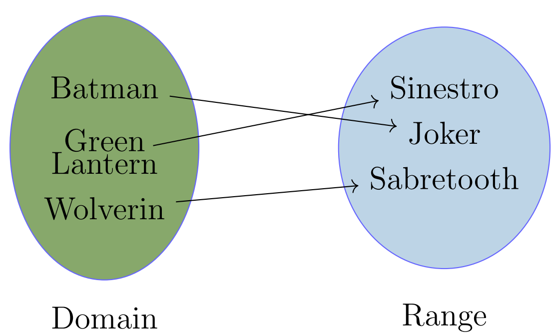
Objective 25: Identify one-to-one functions

Only one-to-one functions have an inverse

Example II.2. *Determine if the relation defines y as a one-to-one function of x*

A $\{(2,1), (4,2), (7,3), (-2,1)\}$

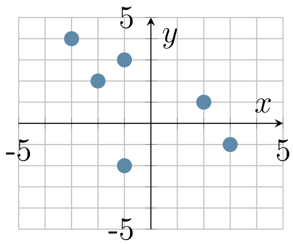
B



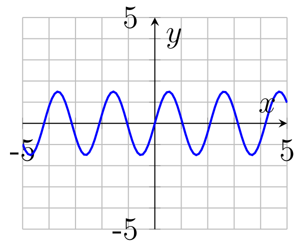
Remark. *A function $y = f(x)$ is a one-to-one function if no horizontal line intersects the graph in more than one place.*

Example II.3. *Determine if the above remark applies to the examples below*

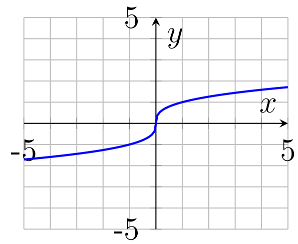
A



B



C



EXAMPLE II.4. *Use the definition of a one-to-one function to determine whether the function is one-to-one. That is, show that $f(a) = f(b)$, then $a = b$*

A $f(x) = 3x + 2$

B $f(x) = 3x^2$

Definition II.4.1: Inverse of a function

Let f be a one-to-one function. Then g is the **inverse** of f if the following conditions are both true:

(1) $(f \circ g)(x) = x$ for all x in the domain of g

(2) $(g \circ f)(x) = x$ for all x in the domain of f

Objective 26: Determine whether two functions are inverses

Given a function $f(x)$ and its inverse $f^{-1}(x)$, then the definition implies that $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(x) = x$

EXAMPLE II.5. *Determine whether the two functions are inverses*

A $f(x) = 3x + 6$ and $g(x) = \frac{1}{3}x - 2$

B $f(x) = \frac{x+1}{5}$ and $g(x) = 5x + 1$

Properties of inverse function

(1) If $f(x)$ and $g(x)$ are inverse functions, the domain of $f(x)$ is the same as the range of $g(x)$

(2) (a, b) is a point on the graph of $y = f(x)$ if and only if (b, a) is a point on the graph of $y = f^{-1}(x)$

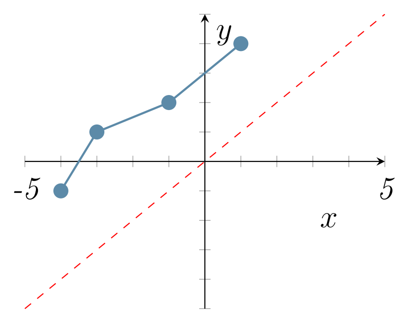
(3) The graph of $y = f^{-1}(x)$ is the reflection of the graph $y = f(x)$ about the line $y = x$

Example II.6. *The graph of a function is given*

A *Find the domain and range of the function.*

B *Find the domain and range of the inverse.*

C *Graph the inverse function.*



Objective 27: Find the inverse of a function

For a one-to-one function defined by $y = f(x)$, follow the given procedure to determine the inverse of the function.

Procedure to find an equation of an inverse of a function

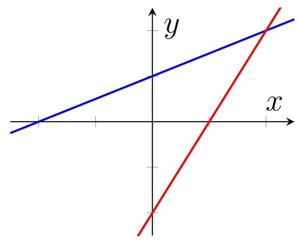
Step 1: Replace $f(x)$ by y

Step 2: Interchange x and y

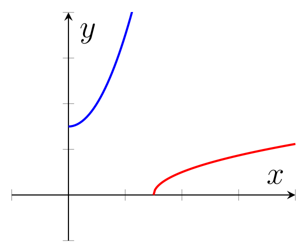
Step 3: Solve for y

Step 4: Replace y by $f^{-1}(x)$

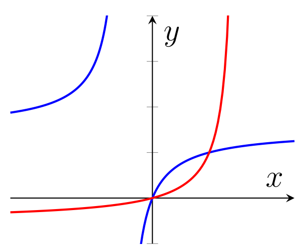
EXAMPLE II.7. *For the one-to-one function $f(x) = 2 + \frac{1}{2}x$, write an equation for the inverse function.*



EXAMPLE II.8. *For the one-to-one function $f(x) = x^2 + 3$ for $x \geq 0$, find the inverse function.*



EXAMPLE II.9. *For the one-to-one function $f(x) = \frac{3x}{x+1}$, find $f^{-1}(x)$.*

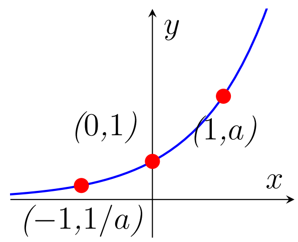


II.10. Exponential Functions (ALEKS 3.2)

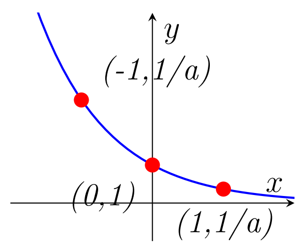
Definition II.10.1: Exponential Function with base b

Let b be a constant real number such that $b > 0$ and $b \neq 1$. Then for any real number x, a function of the form $f(x) = b^x$ is called an **exponential function of base** b

Example II.11. *The graph of $f(x) = a^x$ where $a > 1$ is always increasing. State the domain, range, intercepts and asymptotes.*



Example II.12. *The graph of $f(x) = a^x$ where $0 < a < 1$ is always decreasing. State the domain, range, intercepts and asymptotes.*



Definition II.12.1: Exponential Function with base b

An exponential function is a function of the form $f(x) = Ca^x$ where a is a real number with $a > 0$ and $a \neq 1$. The base a is called the **growth factor** and C the **initial value**.

For an exponential function $f(x) = Ca^x$, the ratio of $f(x+1)$ and $f(x)$ is the constant $\frac{f(x+1)}{f(x)} =$ ____________________________________.

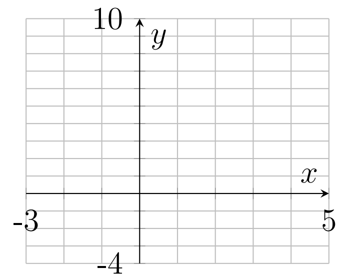
Every exponential function $f(x) = b^x$ passes through three points:____________________

Objective 28: Graphing exponential functions

Instead of variable to a power in, e.g., $y = x^2$, $y = x^3 - 1$, we shall graph variable in the exponent, as in, for example, $y = 2^x$ and $y = 3^{x-1}$

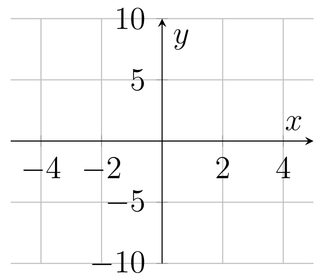
Example II.13. *Graph the function* $f(x) = 2^{-x}$

x	$f(x)$



Example II.14. *Graph* $f(x) = 1 - 2^{x+1}$

x	$f(x)$



Transformations

$$f(x) = ab^{x-h} + k$$

If $a < 0$, reflect across the x-axis Shrink vertically if $0 < \lvert a \rvert < 1$ Stretch vertically if $\lvert a \rvert > 1$	If $k > 0$, shift upward. If $k < 0$, shift downward	If $h > 0$, shift to the right If $h < 0$, shift to the left.

Objective 29: Evaluate the exponential function with base e

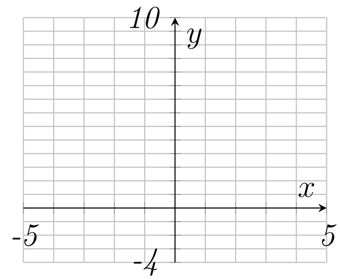
e is a universal constant (like the number π) and an irrational number. $e \approx 2.718281828$. As $x \to \infty$, the expression $\left(1+\frac{1}{x}\right)^x$ approaches a constant value that we call e.

EXAMPLE II.15. *Evaluate the following. Round to 4 decimal places*

A e^2 **B** e^{-3} **C** $e^{\sqrt{7}}$ **D** e^{π}

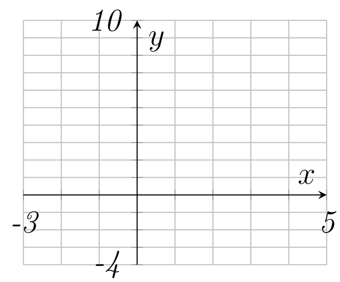
EXAMPLE II.16. *Graph* $f(x) = e^{x+1}$

x	$f(x)$



EXAMPLE II.17. *Graph* $g(x) = -e^x - 3$

x	$f(x)$



Objective 30: Use exponential functions to compute compound interest

Suppose that P dollar in principal is invested (or borrowed) at an annual interest rate r for t years. Then

$I = Prt$	Amount of simple interest I (in \$).
$A = P\left(1+\frac{r}{n}\right)^{nt}$	Amount A (in \$) in the account after t years and n compounding periods per year
$A = Pe^{rt}$	Amount A (in \$) in the account after t years under continuous compounding.

Example II.18. *Suppose that \$15,000 is invested with 2.5% interest under the following compounding options. Determine the amount in the account at the end of 7 years for each option*

A *Compounded annually*

B *Compounded quarterly*

C *Compounded continuously*

Objective 31: Use exponential functions in applications

Example II.19. *Weapon-grade plutonium is composed of approximately 93% plutonium-239 (Pu-239). The half-life of Pu-239 is 24,000 years. In a sample originally containing 0.5 kilograms, the amount left after t years is given by*

$$A(t) = 0.5\left(\frac{1}{2}\right)^{t/24000}$$

Evaluate the function for the given values of t and interpret the meaning in context

A $A(24,000)$

B $A(72,000)$

EXAMPLE II.20. *A 2010 estimate of the population of Mexico is 111 million people with a projected growth rate of 0.994% per year. At this rate, the population* $P(t)$ *(in millions) can be approximated by*

$$P(t) = 111\,(1.00994)^t$$

where t *is the time in years since 2010.*

A *Evaluate* $P(4)$ *and interpret its meaning in the context of this problem.*

B *Evaluate* $P(75)$ *and interpret its meaning in the context of this problem.*

II.21. Logarithmic Functions (ALEKS 3.3)

Objective 32: Convert between logarithmic and exponential forms

The inverse of an exponential function, base b, is a logarithmic function base b

Definition II.21.1: Logarithmic Function

If x and b are positive real numbers such that $b \neq 1$, then $y = \log_b x$ is called the **logarithmic function** with base b where

$$y = \log_b x \text{ is equivalent to } b^y = x$$

"A logarithm is an exponent"

Example II.22. *Write each equation in exponential form*

A $\log_4 16 = 2$

B $\log_5 0.2 = -1$

Example II.23. *Write each equation in logarithmic form*

A $\left(\frac{1}{4}\right)^{-2} = 16$

B $27^{1/3} = 3$

Objective 33: Evaluate logarithmic expressions

To evaluate a logarithmic expression algebraically, write the expression in exponential form and use $a^u = a^v \iff u = v$

Example II.24. *Evaluate without using a calculator*

A $\log_{12} 144$

B $\log_{64} 8$

C $\log_3 \left(\frac{1}{27}\right)$

Definition II.24.1: Common and Natural Logarithmic Function

Common logarithmic function: $y = \log_{10} x$ is written as $y = \log x$

Natural logarithmic function: $y = \log_e x$ is written as $y = \ln x$

EXAMPLE II.25. *Evaluate each expression*

A $\log 100$

B $\log 0.0001$

C $\ln e^{-3}$

D $\ln e^{10}$

Objective 34: Apply basic properties of logarithms

From the definition of a logarithmic function, we have the following basic properties

1 $\log_b 1 = 0$ because $b^0 = 1$

2 $\log_b b = 1$ because $b^1 = b$

3 $\log_b b^x = x$ because $b^x = b$

4 $b^{\log_b x} = x$ because $\log_b x = \log_b x$

Properties **3** and **4** follow from the fact that a logarithmic function is the inverse of an exponential function.

EXAMPLE II.26. *Simplify the expression*

A $\log_{19} 19$

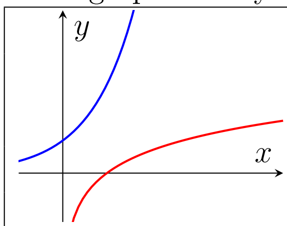
B $e^{\ln(4x)}$

C $\log_{1/8} 1$

D $\log_{30} 30^{10}$

REMARK. *Exponential and logarithmic function are inverses of each other*

Their graphs are symmetric with respect to the line $y = x$.



	$y = b^x$	$y = \log_b x$
For $b > 1$	Domain: $(-\infty, \infty)$	Domain: $(0, \infty)$
$y = \log_b x$	Range: $(0, \infty)$	Range: $(-\infty, \infty)$
$y = b^x$	Horizontal asymptote: $y = 0$	Vertical asymptote: $x = 0$
	Passes through $(0, 1)$	Passes through $(1, 0)$

Transformations of logarithmic graphs

$$f(x) = a \log_b (x - h) + k$$

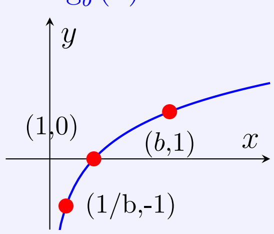
If $a < 0$, reflect across the x-axis		
Shrink vertically if $0 < \lvert a \rvert < 1$	If $k > 0$, shift upward.	If $h > 0$, shift to the right
Stretch vertically if $\lvert a \rvert > 1$	If $k < 0$, shift downward	If $h < 0$, shift to the left.

Objective 35: Graphing logarithmic functions

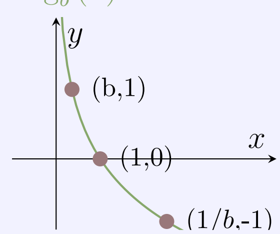
If $b > 1$, the function is always increasing.

If $0 < b < 1$, the function is always decreasing.

$y = \log_b (x)$ with $b > 1$



$y = \log_b (x)$ with $0 < b < 1$



The domain of each function is ____________________

The range of each function is ____________________

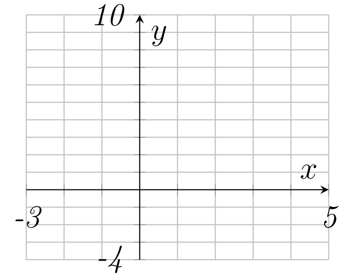
If $b > 1$, the x-intercept is ____________________.

If $0 < b < 1$, the x-intercept is ____________________.

The vertical asymptotes are ____________________.

Example II.27. *Graph* $y = \log_2 (1 + x) + 2$ *Rewrite in exponential form and select values for y first*

x	$f(x)$



Definition II.27.1: Domain of logarithm

The domain of a logarithm function is the set of real numbers that make the argument positive

EXAMPLE II.28. *Give the domain in interval notation. What is the equation for the vertical asymptote?*

A $y = \log_2(x+3)$

B $\log(x^2 - 3x - 4)$

II.29. Properties of Logarithm (ALEKS 3.4)

Objective 36: Apply the product, quotient and power properties of logarithms

Because a logarithm is an exponent, the properties of exponents can be applied to logarithms

Product Property

Let b, x, and y be positive real numbers where $b \neq 1$. Then

$$\log_b (xy) = \log_b x + \log_b y$$

That is, logarithm of a product equals the sum of the logarithms of the factors

Example II.30. *Write the logarithm as a sum. Simplify if possible. Assume all variable expressions represent positive real numbers.*

A $\ln(3(a+b))$

B $\log_2(2xy)$

Quotient Property

Let b, x, and y be positive real numbers where $b \neq 1$. Then

$$\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$$

That is, logarithm of a quotient equals the difference of the logarithms of numerator and the logarithm of the denominator.

Example II.31. *Write the logarithm as a difference. Simplify if possible.*

A $\log_7\left(\frac{z}{49}\right)$

B $\log\left(\frac{100}{x+y}\right)$

Power Property

Let b, x, and p be positive real numbers where $b \neq 1$. Then

$$\log_b\left(x^p\right) = p\log_b x$$

That is, the logarithm of an argument raised to a power equals the power times the logarithm.

Example II.32. *Apply the power property of logarithm on the following*

A $\ln\left(x^5\right)$

B $\log\sqrt[3]{x^2}$

Objective 37: Write a logarithmic expression in expanded form

In some applications of algebra and calculus, the expanded form is preferred. In expanded form, the arguments are "simple".

Example II.33. *Write the expression as the sum or difference of logarithms*

A $\ln\left(\frac{\sqrt[3]{x^2}}{w+z}\right)$

B $\log_8\left(\frac{64x^2y}{3zw^3}\right)$

Objective 38: Write a logarithmic expression as a single logarithm

In condensed form, the sum and difference of logarithms is simplified to a single logarithm term with a "complex" argument

Example II.34. *Write the logarithmic expression as a single logarithm and simplify as much as possible.*

A $-2\log_5{(x+5)} + \log_5 x + \log_5{(x^2-25)}$

B $2\left[\ln x - \ln{(x^2)}\right] + \frac{2}{3}\ln z^9 - \ln z$

Objective 39: Apply the change-of-base formula

The change-of-base formula converts a logarithm of one base to a ratio of logarithms of a different base. When using a calculator to approximate values, we convert to log base 10 or log base e.

Change of base formula.

Let a and b be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then, for any positive real number x,

$$\log_b x = \frac{\log_a x}{\log_a b}$$

In particular,

$$\log_b x = \frac{\log x}{\log b} \text{ or } \frac{\ln x}{\ln b}$$

Example II.35. *Use the change-of-base formula to rewrite using common and natural logarithms. Use a scientific calculator to approximate the logarithm to 4 decimal places*

A $\ln_7 15$

B $\log_5 0.3$

II.36. Exponential and Logarithmic Equations (ALEKS 3.5)

Objective 40: Solve exponential equations

Some exponential equations can be solved using the **equivalence property** of exponential expressions: if b, x and y are real numbers with $b > 0$ and $b \neq 1$, then $b^x = b^y \implies x = y$

Example II.37. *Solve the following exponential equations using the equivalence property*

A $5^{3x-1} = 125$

B $\frac{1}{4} = 8^x$

C $8^{x+3} = 4^{5-2x}$

Most exponential equations cannot be rewritten to have matching bases. To solve these equations, isolate the exponential expression, take a logarithm on both sides, and then apply the power property of logarithms.

Example II.38. *Solve the following exponential equations*

A $2^x = 7$

B $3^{-0.02x} = 0.07$

C $4 + 3e^{2x} = 11$

D $2^{3x-2} = 3^{2x+1}$

Example II.39. *Use u-substitution to solve exponential equations that are quadratic in form. Solve* $e^{2x} - 8e^x + 12 = 0$

Objective 41: Solve logarithmic equations

Note the difference between these two equations:

1 $\log(3x-4) - \log(x+1) = \log 2$ **2** $\log_3(x+6) - \log_3(x-2) = 2$

In the first equation, all terms have a logarithm. The second equation is a mix of logarithmic and constant terms. These two equations require different styles of solution. For solving equations of the first type, we will use the equivalence property of logarithmic expressions: if b, x and y are real numbers with $b > 0$ and $b \neq 1$, then $\log_b x = \log_b y \implies x = y$

Example II.40. *Solve the following*

A $\log_2(5x-1) = \log_2(2x+17)$

B $\log(x-7) = \log(5-x)$

C $\log(3x-4) - \log(x+1) = \log 2$

D $\ln(x+3) + \ln x = \ln 10$

E $2\log_3 x = \log_3 2 + \log_3(3x-4)$

Objective 42: Solve logarithmic equations with a mix of logarithmic and constant terms.

To solve equations that are a mix of logarithms and constants, collect all logarithms together, then rewrite the equation in exponential form:

$$\log_b x = k \quad \Longrightarrow \quad x = b^k$$

Example II.41. *Solve the following logarithmic equations*

A $\log_5(3x+1) = 2$

B $\log_3(x+6) - \log_3(x-2) = 2$

C $\log(x+15) = 2 - \log x$

D $\log_4(x^2 - 9) - 3 = \log_4(x+3)$

Objective 43: Use exponential and logarithmic equations in applications

EXAMPLE II.42. *If \$20,000 is invested in an account earning 2.5% interest compounded continuously, determine how long it will take the money to grow to \$45,000. Round to the nearest year. Use the model* $A = Pe^{rt}$

EXAMPLE II.43. *The following formula relates the energy* E *(in joules) released by an earthquake of magnitude* M*, where* $M > 5$.

$$\log E = 5.24 + 1.44$$

What is the energy released by a magnitude 6 and a magnitude 7 earthquake?

As a comparison, a 1 megaton nuclear weapon would have an energy of 4.18×10^{15} joules and the eruption of the volcano Krakatoa in 1883 produced an energy release of 2.09×10^{19} joules.

II.44. Exponential Growth and Decay (ALEKS 3.6)

Objective 44: SolVe literal equations for a specified variable

Example II.45. *Solve for D in $S = 23.5 + 2\log D$*

Example II.46. *Solve for x in $3e^{nx} - t = 4.2$*

Objective 45: Create models for exponential growth and decay

Let y be a variable changing exponentially with respect to t, and let y_0 represent the initial value of y when $t = 0$.

For a positive constant k	For a negative constant k
$y = y_0e^{kt}$	$y = y_0e^{kt}$
is a model for	is a model for
exponential growth	exponential decay

Example II.47. *Suppose that at 19 years old, you win \$100,000 playing the lottery. If you would like to have \$1,000,000 when you retire at age 67, determine the average rate of return needed under continuous compounding*

Example II.48. *According to a study published in 2012, Burmese pythons are becoming the new top predator in the Florida Evergrades. Not many animals in the Everglades can stand against a python that can grow to 23 feet and weigh over 200 pounds.*

A *Twenty pythons were recorded as captured or killed in 1995. By 2009, that number had increased to 378. Write a function of the form* $P(t) = P_0 e^{rt}$ *to represent the number of pythongs captured or killed* t *years after 1995*

B *Use the model from A to approximate the number of pythongs captured or killed in 2017*

C *If the ratio of pythons living in the wild to pythons captured or killed is approximately 100:1, how many pythons were estimated to be in the Everglades in 2012?*

Example II.49. *A sample collected from cave paintings on an archeological site in France shows that only 2% of the carbon-14 still remainds. How old is the sample? Round to the nearest year. Use the following model for radiocarbon dating*

$$Q(t) = Q_0 e^{-0.000121t}$$

where Q_0 *is the original quantity of carbon-14*

Students - Check your understanding.

CYU 10. *Simplify the expression*

A $8^{\log_8 5}$

B $10^{\log 14}$

TOPIC III

Systems of Equations, Matrices and Determinants

III.1. Solve Systems of Linear Equations(ALEKS 8.1 & 8.2)

Objective 46: Identify solutions to systems of linear equations in two variables

A solution to a system of equations in two variables is the ordered pair that is a solution to each individual equation.

Example III.2. *Determine if the ordered pair is a solution to the system*

A

$$\begin{aligned} 2x + 6y& \\ x + 5y &= 3 \end{aligned}$$

$(-3, 5)$

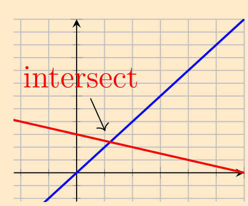
B

$$\begin{aligned} 5x + 10y &= 5 \\ 22x - 6y &= -3 \end{aligned}$$

$(0, 1/2)$

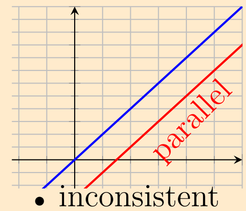
Solutions to systems of linear equations in two variables by graphing

One unique solution: A system of linear equations that represents intersecting lines has exactly one solution.



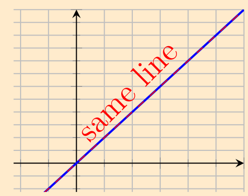
- one solution
- lines intersect
- consistent
- independent

No solution: If a system of linear equations represents parallel lines, the lines do not intersect, and the system has no solution. In such a case, we say that the system is inconsistent.



- inconsistent
- lines parallel
- no solution

Infinitely many solutions: If a system of linear equations represent the same line, then all points on the common line satisfy each equation. Therefore, the system has infinitely many solutions. In such a case, we say that the equations are dependent.



- lines are same
- infinite solutions
- consistent
- dependent

Objective 47: Solve systems of linear equations in two variables

There are two algebraic methods:

Substitution: Isolate one variable. Substitute into the *other* equation. Solve the resulting equation for one variable.

Addition: Write both equations as $Ax + By = C$. Create opposite coefficients for one variable. Add and solve for one variable.

In either method, substitute known value back into one original equation to find the other value.

Example III.3. *Consider the system*

$$\begin{aligned} 2x + 3y &= 10 \\ 5x - y &= -26 \end{aligned}$$

A *Solve the system by the substitution method*

B *Solve the system by the addition method*

Example III.4. *Solve*

$$\begin{aligned} x + \frac{5}{3}y &= \frac{1}{3} \\ \frac{3}{2}x + \frac{5}{2}y &= 2 \end{aligned}$$

Objective 48: Identify solutions to a system of linear equations in three variables

A solution to a system of linear equations in three variables is an ordered triple (x, y, z) that satisfies each equation in the system.

EXAMPLE III.5. *Determine if the ordered triple* $(0, -2, 5)$ *is a solution to the system*

$$\begin{aligned} x - 4y + z &= 13 \\ 3x + y - z &= -7 \\ 2x + 3y + z &= -1 \end{aligned}$$

Objective 49: Solve systems of linear equation in three variables

Step 1: Choose a pair of equations and eliminate one of the variables by using the addition method.

Step 2: Choose a different pair of equations and eliminate the *same* variable.

Step 3: You should now have two equations in two variables.

Step 4: Solve this system by using the substitution or addition method.

Step 5: Substitute the values of the variables found above into any of the three original equations that contain the third variable and solve for the third variable.

Solutions to systems of linear equations in three variables by graphing

One unique solution:	This is a system of linear equations that represents intersecting planes with exactly one point common to all three planes. In such a case, we say that the system is consistent.	3d graph
Infinitely many solutions:	If a system of linear equations represents planes intersecting at a line, the system has infinitely many solutions. Such a system is also consistent.	3d graph
No solution:	If a system of linear equations represent different planes, no three of which intersect at a common point or common line, then the system is inconsistent and has no solution.	3d graph

Example III.6. *Solve*

$$\begin{aligned} x - \frac{1}{2}y + \frac{1}{3}z &= 5 \\ x - y - z &= -1 \\ 6x + z &= -3y \end{aligned}$$

Example III.7. *Solve*

$$\begin{aligned} y &= 3z - 2x - 3 \\ 2(2x - 3z) + 2y &= -6 \\ 4z + 3x - 2y &= 1 \end{aligned}$$

Example III.8. *Solve*

$$\begin{aligned} 3x + 2y + z &= 3 \\ x - 3y + z &= 4 \\ -6x - 4y - 2z &= 1 \end{aligned}$$

Students - Check your understanding.

CYU 11. *Solve*

$$\begin{aligned} 5x + 0.3y &= 1.4 \\ 0.2x - 0.1y &= 0.28 \end{aligned}$$

CYU 12. *Solve*

$$\begin{aligned} 2x + 4y &= 4 \\ y &= 5x - \frac{7}{4} \end{aligned}$$

CYU 13. *Solve*

$$\begin{aligned} x + y &= 19 \\ 2x - z &= 4 \\ x + y + z &= 29 \end{aligned}$$

III.9. Solve Systems Using Matrices (ALEKS 9.1 & 9.2)

Definition III.9.1: Matrix

A **matrix** is a rectangular array of numbers with m rows and n columns.

Objective 50: Write an Augmented Matrix

A matrix can be used to represent a system of linear equations written in standard form. To do so, we extract the coefficients of each term in the equation to form an augmented matrix. A bar within the augmented matrix separates the coefficients of the variable terms in the equations from the constant terms.

Example III.10. *Write an augmented matrix for each system*

A $$\begin{aligned} 3x - y &= 10 \\ 5y &= 12 \end{aligned}$$

B $$\begin{aligned} x + z &= -1 \\ y + 2x &= 5 \\ z - y &= 3 \end{aligned}$$

Example III.11. *Write a system of linear equations represented by the augmented matrix*

A $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0.05 \\ 0 & 0 & 1 & 12 \end{array}\right]$$

B $$\left[\begin{array}{cc|c} -8 & 7 & 113 \\ 2 & 0 & \frac{1}{3} \end{array}\right]$$

Objective 51: Use elementary row operations

Recall the steps we used to solve linear systems algebraically. For matrices, those steps are known as row operations.

Elementary Row Operations

1. Interchange two rows
$$\left[\begin{array}{cc|c} 3 & 2 & -5 \\ 1 & -2 & 10 \end{array}\right] \xrightarrow[\text{interchange rows 1 and 2}]{R_1 \Longleftrightarrow R_2}$$

2. Multiply a row by a nonzero constant
$$\left[\begin{array}{cc|c} 3 & 2 & -5 \\ 1 & -2 & 10 \end{array}\right] \xrightarrow[\substack{\text{multiply row 1 by 3 and replace} \\ \text{row 1 by result}}]{3R_1 \longrightarrow R_1}$$

3. Add a multiple of one row to another row
$$\left[\begin{array}{cc|c} 3 & 2 & -5 \\ 1 & -2 & 10 \end{array}\right] \xrightarrow[\substack{\text{multiply row 1 by 2, add it to row 2} \\ \text{and then replace row 2 by result}}]{2R_1 + R_2 \longrightarrow R_2}$$

Example III.12. *Perform the following elementary row operations*

A $$\left[\begin{array}{cc|c} -2 & 4 & 10 \\ 1 & 5 & 3 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 \to R_1} \qquad\qquad \xrightarrow{R_1 + R_2 \to R_2}$$

B $$\left[\begin{array}{ccc|c} 12 & 0 & -6 & 3 \\ 2 & -8 & 4 & -18 \\ 2 & 1 & 7 & 0 \end{array}\right] \xrightarrow{-R_3 + R_2 \to R_2} \qquad\qquad \xrightarrow{-\frac{1}{9}R_2 \to R_2}$$

Objective 52: Use Gaussian Elimination and Gauss-Jordan Elimination

If we perform repeated row operations, we can form an augmented matrix that represents a system of equations that is easier to solve than the original system. In particular, it is easy to solve a system whose augmented matrix is in row-echelon form or reduced row-echelon form.

Definition III.12.1: Row-echelon form

(1) Any rows consisting entirely of zeros are at the bottom of the matrix.

(2) For all other rows, the first nonzero entry is 1. This is called the leading 1.

(3) The leading 1 in each nonzero row is to the right of the leading 1 in the row immediately above

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & \frac{3}{5} \\ 0 & 1 & 4 & -8 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 2 & 12 & 70 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Definition III.12.2: Reduced Row-echelon form

(1) The matrix is in row-echelon form

(2) All entries above the leading 1 are zeros

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 12.6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3.8 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 14 & 28 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example III.13. *Which of the following are in row echelon form? Which of the following are in reduced row echelon form?*

		REF?	RREF?
A	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$		
B	$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$		

C $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		
D $\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$		
E $\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$		

Remark. *Note that a matrix in reduced row echelon form is also in echelon form, but not necessarily the other way around.*

After writing an augmented matrix in row-echelon form, the corresponding system of linear equations can be solved using back substitution. This method is called Gaussian Elimination.

To solve a system of linear equations using Gaussian elimination,

(1) Write the augmented matrix for the system.
(2) Use elementary row operations to write the augmented matrix in row-echelon form. Begin by getting the entry in row 1 column 1 to be 1, with zeros under it.
(3) Use back substitution to solve the resulting system of equations.

Example III.14. *Solve the systems using Gaussian elimination*

A
$$\begin{aligned} 2x + 3y &= 7 \\ 3(x+1) &= 5y + 4 \end{aligned}$$

$$\mathbf{B} \quad \begin{aligned} 2x - 6y - z &= 7 \\ x + 2y + z &= 0 \\ x + 4y + 2z &= -3 \end{aligned}$$

If we write an augmented matrix in *reduced* row-echelon form, we can solve the corresponding system of equations by inspection. This is called the Gauss-Jordan elimination method.

EXAMPLE III.15. *Continue to perform row operations on the following matrix until it is reduced row-echelon form. Then determine the solution by inspection.*

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 2 & -8 \end{array}\right]$$

Objective 53: Identify inconsistent systems

A system of equations may have <u>no solution</u>. Such a system is said to be inconsistent.

An inconsistent system will reduce to a contradiction. This means that the planes do not intersect *all* intersect.

EXAMPLE III.16. *Write a system of linear equations represented by the augmented matrix and solve the system*

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -18 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

Objective 54: Solve systems with dependent equation

A system of linear equations may have infinitely many solutions. In this case, the equations are said to be dependent. For a system of three variables, this means that the planes intersect in a common line in space, or the three planes all coincide. A system of linear equations must have at least the same number of unique equations as it has variables to have a unique solution.

EXAMPLE III.17. *Solve the system*

$$\begin{aligned} x - 2y + z &= 5 \\ 3x + y - 4z &= 1 \\ 2x - 4y + 2z &= 10 \end{aligned}$$

Students - Check your understanding.

CYU 14. *Write an augmented matrix for the system*

$$\begin{aligned} 2x + 3(x + y) &= 1 \\ 6(x - y) &= 14 \end{aligned}$$

CYU 15. *Solve the system*

$$\begin{aligned} x - 6y + 8z &= 7 \\ 2x + 3y + z &= -1 \\ 5x + 9y - 17z &= -4 \end{aligned}$$

CYU 16. *Write a system of linear equations represented by the augmented matrix and solve the system*

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -18 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

CYU 17. *Solve*

$$\begin{aligned} -15x - 5y + 40z &= -30 \\ 2x + y - 8z &= 6 \\ 2x + \frac{2}{3}y - \frac{16}{3}z &= 4 \end{aligned}$$

The system is equivalent to a single equation. This system represents 3 coincident planes. The solution set is the set of all ordered tuples that satisfy the equation $\{(x, y, z) : -3x - y + 8z = -6\}$

CYU 18. *Solve the system*

$$\begin{aligned} x - 5y - 5z &= 1 \\ 6x - 30y - z &= 6 \\ 2x - y - 10z &= 2 \end{aligned}$$

III.18. Matrix Algebra (ALEKS 9.3)

Objective 55: Determine the order of a matrix

The order of a matrix is determined by the number of rows and number of columns. A matrix with m rows and n columns is an $m \times n$ matrix.

EXAMPLE III.19. *Determine the order of the given matrices*

A $\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$ **B** $\begin{bmatrix} 15 & 0 & 3 & \sqrt{5} \end{bmatrix}$ **C** $\begin{bmatrix} 0 \\ x \\ e \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 & 5 \end{bmatrix}$

A matrix can be represented generically by

$$[a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

The notation a_{ij} represents the element in the ith row, jth column. For example, a_{23} is represents the element in the 2nd row, 3rd column.

EXAMPLE III.20. *Determine the value of the given element of the matrix*

$$A = [a_{ij}] = \begin{bmatrix} -5 & \sqrt{3} & -\frac{3}{8} & 0 \\ 0 & \pi & 115 & 1 \end{bmatrix}$$

A $a_{12} =$ ________ **B** $a_{22} =$ ________ **C** $a_{14} =$ ________

Objective 56: Add and subtract matrices

To add or subtract two matrices, the matrices must have the same order, and the sum or difference is found by adding or subtracting the corresponding elements.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ by m by n matrices. Then,

$$\begin{aligned} A + B &= [a_{ij} + b_{ij}] \text{ for } i = 1, 2, ..., m \text{ and } j = 1, 2, ..., n. \\ A - B &= [a_{ij} - b_{ij}] \text{ for } i = 1, 2, ..., m \text{ and } j = 1, 2, ..., n. \end{aligned}$$

Matrix addition is commutative. That is, $A+B = B+A$. However, matrix subtraction is not: $A - B \neq B - A$.

EXAMPLE III.21. *Solve the following*

A $\begin{bmatrix} 1 & -8 \\ 5 & 20 \end{bmatrix} - \begin{bmatrix} 2.3 & 5 \\ 0 & -12 \end{bmatrix}$

B $\begin{bmatrix} 10 & 0 & 18 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$

Definition III.21.1: Additive inverse and Zero Matrix

The additive inverse of a matrix is found by taking the opposite of each element in the matrix: given $A = [a_{ij}]$, the additive inverse of A, denoted by $-A$, is $-A = [-a_{ij}]$. A matrix in which all elements are zero is called a zero matrix, and is denoted by O. The sum of a matrix A and its additive inverse $-A$ is the zero matrix of the same order.

EXAMPLE III.22. *Given*

$$A = \begin{bmatrix} -5 & 8 & 10 \\ 3 & 0 & -\frac{1}{3} \end{bmatrix}$$

find the additive inverse $-A$.

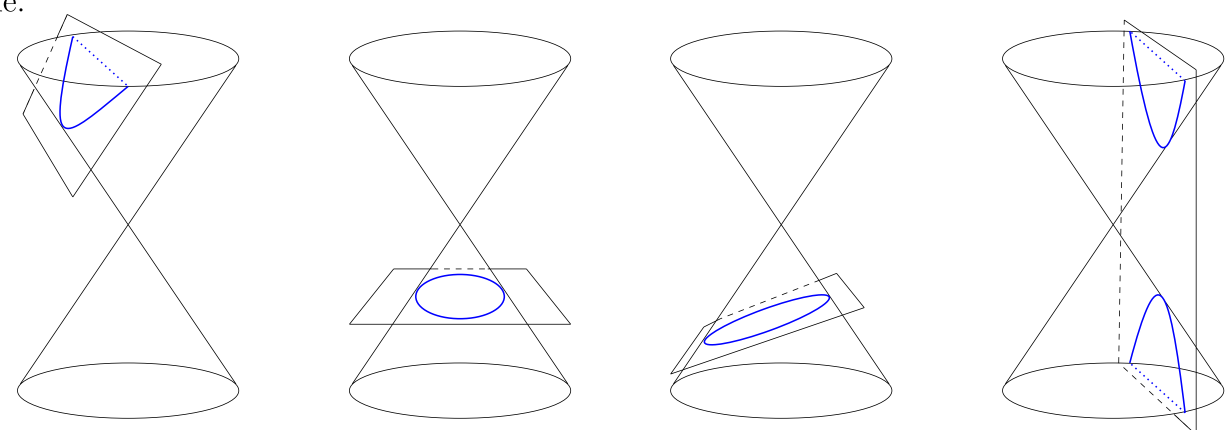
III.23. Determinants (ALEKS 9.5)

Students - Check your understanding.

TOPIC IV

Conic Sections

Conics, an abbreviation for conic sections, are cross sections that result from the intersection of a right circular cone and plane. **Circles** are when the plane is perpendicular to the axis of the cone when it intersects. **Ellipses** are when the plane is tilted slightly when it intersects the cone. **Parabolas** are when the plane is tiled farther so that it is parallel to one generator. **Hyperbolas** are when the plane intersects both parts (called nappes) of the cone.



(a) parabola (b) circle (c) ellipse (d) hyperbola

We will look into each type of conic, starting with parabolas

IV.1. The Parabola (ALEKS 10.3)

IV.2. The Ellipse (ALEKS 10.1)

IV.3. The Hyperbola (ALEKS 10.2)

Students - Check your understanding.

TOPIC V

Sequences and Series

V.1. Sequences and Series (ALEKS 11.1)

Definition V.1.1: Sequence

A **sequence** is a function whose domain is the set of positive integers and whose range is the set of all real numbers.

Objective 57: Write terms of a sequence from the nth term

A sequence is often thought of as a list of numbers where the order is important and each number is called a term. The default variable for a sequence is n where $n = 1, 2, 3, ...$

Vocabulary

Finite sequence	A function whose domain is the set of first n positive integers
Infinite sequence	A function whose domain is the set of positive integers
Terms of a sequence	Denoted by the letter a with a subscript representing the term number $$\{a_n\} = a_1, a_2, a_3, ...$$

Example V.2. *Write the first four terms of the sequence defined by the nth term of* $a_n = n^2 + 5$

Example V.3. *Write the first four terms of the sequence defined by the nth term of* $a_n = 3\left(-\frac{1}{3}\right)^n$

Notice that the terms of the sequence in **Example V.3** alternate in signs. Such a sequence is called an alternating sequence.

The alternation in signs can be represented in the nth term by a factor of -1 raised to a variable power that alternates between an even and odd integer. For example,

$a_n = (-1)^n$ odd-numbered terms negative and even-numbered terms positive

$b_n = (-1)^{n+1}$ odd-numbered terms positive and even-numbered terms negative

Objective 58: Write terms of a sequence defined recursively

Some sequences use earlier terms to define later terms.

Definition V.3.1: Recursion Formula

A **recursion formula** defines the nth term of a sequence as a function of one or more terms preceding it.

Example V.4. *Write the first four terms of the sequence defined by* $a_1 = 3$ *and* $a_n = 4a_{n-1} - 2$

Example V.5. *Find the next five terms of the Fibonacci sequence with* $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-2} + a_{n-1}$

Objective 59: Use factorial notation

We use factorial notation to denote the product of the first n positive integers.

Definition V.5.1: Factorial

Let n be a positive integer. The expression $n! = (n)(n-1)(n-2)\dots(2)(1)$. For $n = 0$, we define $0! = 1$

Example V.6. *Evaluate the following expressions*

A $5!$

B $\frac{6!}{4!}$

C $\frac{9!}{4!5!}$

D $\frac{n!}{(n-1)!}$

Example V.7. *Given the sequence defined by* $a_n = \frac{(3n)!}{n^3}$, *find* a_3.

Objective 60: Use summation notation, represented by the Greek letter Σ (sigma)

he sum S of the infinite sequence $\{a_n\} = a_1, a_2, a_3, \dots$ is called an infinite sum and is represented by

$$S = a_1 + a_2 + a_3 + \dots$$

The sum of the first n terms of the sequence is called the nth partial sum of the sequence and is denoted by S_n

The nth partial sum of a sequence is a finite series.

Often, we want to find the sum of the first n terms of a sequence $\{a_n\}$. Rather than writing down all the terms, we use the summation notation where

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + ... + a_n$$

- The lower limit of the sum is 1.
- The upper limit of the sum is n.
- The index of this sum is i.

The sum of *all* terms in the sequence is an infinite series and is given by

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + ...$$

EXAMPLE V.8. *Write the terms for each series and find the sum*

A $\sum_{k=1}^{3} (3k - 8)$

B $\sum_{i=1}^{i} \frac{2i}{i^2}$

C $\sum_{j=3}^{5} (-1)^{j+1} (3j)$

D $\sum_{n=1}^{4} 10$

Example V.9. *Note that the same sum may be written in many different ways*

$$\sum_{i=1}^{4} i^2 \qquad \sum_{j=0}^{3} (j+1)^2 \qquad \sum_{k=2}^{5} (k-1)^2$$

Example V.10. *Write each sum using summation notation*

A $\frac{2}{5} + \frac{4}{5} + \frac{8}{5} + \frac{16}{5} + \frac{32}{5}$

B $\frac{1}{2} + 1 + 2 + 4 + 8$

C $-\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81}$

Properties of Summation

If $\{a_n\}$ and $\{b_n\}$ are sequences, and c is a real number, then:

(1) $\sum_{i=1}^{n} c = cn$

(2) $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

(3) $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

V.11. Arithmetic Sequences and Series (ALEKS 11.2)

Objective 61: Identify specific and general terms of an arithmetic sequence

When the difference between successive terms is constant, the sequence is called arithmetic.

Definition V.11.1: Arithmetic sequence

An arithmetic sequence is defined recursively as $a_1 = a$; $a_n = a_{n-1} + d$, where a and d real numbers.

Example V.12. *Determine whether the sequence is arithmetic. If so, identify the common difference*

A $1, 3, 9, 27, 81, \ldots$

B $4, 9, 14, 19, 24, \ldots$

Example V.13. *Write the first four terms of an arithmetic sequence with first* -2 *and common difference* -3

Definition V.13.1: nth Term of an Arithmetic Sequence (Explicit Rule)

The nth **term of an arithmetic sequence** is given by

$$a_n = a_1 + (n-1)\,d$$

where a_1 is the first term of the sequence and d is the common difference.

Example V.14. *Find the twenty-third term of the arithmetic sequence* $-8, -3, 2, 7, \ldots$

Example V.15. *Find the 40th term of an arithmetic sequence in which* $a_1 = 12$ *and* $a_{11} = -38$

Example V.16. *Find the number of terms in the finite arithmetic sequence* $3, 8, 13, ..., 98$

Objective 62: Evaluate a finite arithmetic series

Carl Friedreich Gauss recognised patterns of arithmetic sums as an elementary student that led to the following sum formula

nth partial sum of an arithmetic sequence

The sum S_n of the first n terms of an arithmetic sequence is given by

$$S_n = \frac{n}{2}(a_1 + a_n)$$

where a_1 is the first term of the sequence and a_n is the nth term.

Example V.17. *Find the sum of the first 25 terms of the sequence* $3, 7, 11, 15, ...$

Example V.18. *Find the sum* $\sum_{k=1}^{22}(8 - 5k)$

Students - Check your understanding.

CYU 19. *Edward has a savings plan. He starts by putting one dime in his piggy bank and plans to add a quarter to it each week. Write the nth term of a sequence defining the amount of money Edward has in his "bank".*

V.19. Geometric Sequences and Series (ALEKS 11.3)

Objective 63: Identify specific and general terms of a geometric sequence

When the ratio of successive terms in a sequence is constant, the sequence is called geometric.

Definition V.19.1: Geometric Sequence

A **geometric sequence** may be defined recursively as $a_1 = a$, $a_n = ra_{n-1}$ where $a_1 = a$ and r are real numbers.

Theorem V.19.1: (Explicit Rule) nth term of a geometric sequence

For a geometric sequence $\{a_n\}$ whose first term is a_1 and whose common ratio is r, the nth term is determined by the formula

$$a_n = a_1 r^{n-1}$$

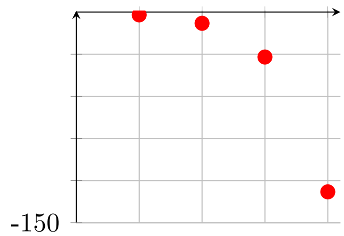
Example V.20. *Determine whether the sequence is geometric. If so, identify the common ratio.*

A $2, 6, 18, 54, ...$

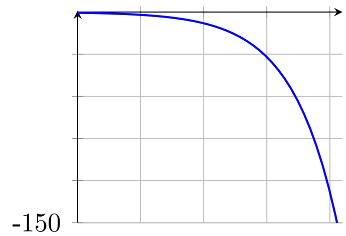
B $2, 6, 10, 14, ...$

Example V.21. *Write the first four terms of the geometric sequence with $a_1 = -2$ and* $r = 4$

$a_n = -2(4)^{n-1}$



$f(x) = -2\,(4)^{x-1}$



Example V.22. *Write a formula for the nth term of the geometric sequence* $5, 4, \frac{16}{5}, \frac{64}{125}, \ldots$

Example V.23. *Find the sixth term of a geometric sequence where* $a_1 = 10$ *and* $a_2 = 12$

Example V.24. *Find* r*,* a_1 *and an expression for the nth term of a geometric sequence where* $a_4 = -240$ *and* $a_7 = 1920$

Objective 64: Evaluate finite geometric series

The sum S_n of the first n terms in a geometric sequence is called a finite geometric series. That is,

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a_1 r^{k-1} = a_1 + a_2 + \ldots + a_n$$

Theorem V.24.1: Sum of the first n terms of a geometric sequence

Let $\{a_n\}$ be a geometric sequence with first term a_1 and common ratio r. The sum S_n of the first n terms of $\{a_n\}$ is

$$S_n = \frac{a_1}{1-r}\left(1 - r^n\right)$$

Example V.25. *Find the sum* $\sum_{i=1}^{8} 3\left(-\frac{1}{3}\right)^{i-1}$

Example V.26. *Find the sum of the finite geometric series* $3, 6, 12, ..., 1536$

Objective 65: Evaluate infinite geometric series

In an infinite geometric series, the sum may converge to a value or it may diverge.

Definition V.26.1: Geometric Series

A **geometric series** is an infinite sum of a geometric sequence.

The notation in this case is

$$S_\infty = \sum_{k=1}^{\infty} a_1 r^{k-1} = a_1 + a_1 r + a_1 r^2 + ... + a_1 r^{n-1} + a_1 r^n + ...$$

Theorem V.26.1: Convergence of an infinite geometric series

If $-1 < r < 1$, the infinite geometric series

$$\sum_{k=1}^{\infty} a_1 r^{k-1}$$

converges. Its sum is $S_\infty = \frac{a_1}{1-r}$

REMARK. *A series that does not converge is called a divergent series.*

When $r \geq 1$ or $r < -1$, then the sum does not exist. We say that the series diverges.

EXAMPLE V.27. *Find the sum, if possible*

A $\sum_{k=1}^{\infty} -6\left(-\frac{3}{5}\right)^{k-1}$

B $4, -6, 9, -\frac{27}{2}, \ldots$

Students - Check your understanding.

CYU 20. *Write* $0.04\overline{5}$ *as a fraction.*

V.28. Mathematical Induction (ALEKS 11.4)

We have studied infinite sequences of numbers. Now we consider an infinite sequence of propositions. In mathematics, a proposition is a statement that has a truth value – it is either true or false.

Objective 66: Prove a statement using mathematical induction

This technique enables us to prove the validity of a true statement over the set of positive integers

Some sequences may be neither arithmetic nor geometric. Therefore, we have no formula readily available to evaluate the sum of the first n terms. We might consider evaluating the nth partial sum for several values of n to determine if a pattern exists.

For example, if $P_n = 1 + 2 + 3 + 4 + ... + n$, what is a simpler way to write this?

Note that $P_1 = 1$, $P_2 = 1 + 2 = 3$, $P_3 = 1 + 2 + 3 = 6$ and $P_4 = 1 + 2 + 3 + 4 = 10$. We speculate that $P_n = \frac{n(n+1)}{2}$. Note that $P_1 = \frac{1(1+1)}{2} = 1$, $P_2 = 3 = \frac{2(2+1)}{2}$ and $P_3 = 6 = \frac{3(3+1)}{2}$.

How do we prove that our hypothesis is true?

Principle of Mathematical Induction

Two conditions must be satisfies to prove a statement P_n is true for every positive integer n :

(1) **Basis Step:** Show the statement P_n is true for $n = 1$, and

(2) **Induction Step:** If the statement P_n is true for $n = k$, it is true for the next natural number $n = k + 1$.

In other words, in (2), we assume P_k is true and show P_{k+1} is true (using P_k) to conclude P_n is true for all positive integers.

EXAMPLE V.29. *Use mathematical induction to prove that*
$P_n = 1 + 2 + 3 + 4 + ... + n = \frac{n(n+1)}{2}$.

Part 1. Show that P_1 is true
Part 2. Assume that P_k is true and show that P_{k+1} is true.

Mathematical induction can be used to prove summation formulas involving powers of the first n positive integers.

Sums of Powers

Let n represent a positive integer

1. $\sum_{i=1}^{n} 1 = n$

2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

EXAMPLE V.30. *Use mathematical induction to prove that*
$1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$

EXAMPLE V.31. *Use the principle of mathematical induction to show that the given statement is true for all natural numbers n.*

$$\frac{1}{7 \cdot 8} + \frac{1}{8 \cdot 9} + \frac{1}{9 \cdot 10} + ... + \frac{1}{(n+6) \cdot (n+7)} = \frac{n}{7\,(n+7)}$$

Students - Check your understanding.

CYU 21. *Use mathematical induction to prove* $P_n = 2 + 4 + 6 + 8 + ... + 2n = n\,(n+1)$

CYU 22. *Use mathematical induction to prove that*

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+...+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$$

CYU 23. *Use mathematical induction to show that the given statement is true for all natural numbers n:*

$$P_n : 4+8+12+...+4n=2n\,(n+1)$$

V.32. The Binomial Theorem (ALEKS 11.5)

Objective 67: Determine binomial coefficients

The coefficients for the expression of $(a+b)^n$ follow a triangular array of numbers called Pascal's triangle. Each row begins and ends with a 1, and each entry in between is the sum of the two diagonal entries from the row above.

An expression such as $a+b$ is called a binomial. When we square a binomial, we have an expression that is called a binomial expansion. We can expand several powers of $a+b$ and note patterns.

$$\begin{aligned}
(a+b)^0 &= 1\\
(a+b)^1 &= 1a+1b\\
(a+b)^2 &= 1a^2+2ab+1b^2\\
(a+b)^3 &= 1a^3+3a^2b+3ab^2+1b^3\\
(a+b)^4 &= 1a^4+4a^3b+6a^2b^2+4ab^3+1b^4\\
(a+b)^5 &= 1a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+1b^5
\end{aligned}$$

Notice the patterns for each binomial expansion, $(a+b)^n$

(1) Each expansion begins with a^n and ends with b^n.
(2) From left to right, the powers of a are decreasing by 1, while the powers of b are increasing by 1
(3) The number of terms in each expansion is $n+1$
(4) The degree of each monomial in the expansion is n
(5) The coefficients are symmetrical
(6) It is reasonable to conjecture that the expansion of $(a+b)^n$ would look like this:

$$(a+b)^n = a^n + \boxed{n}a^{n-1}b + \boxed{?}a^{n-2}b^2 + ... + \boxed{n}ab^{n-1} + b^n$$

All of this information can be represented in Pascal's Triangle:

§ V.32. The Binomial Theorem (ALEKS 11.5)

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

EXAMPLE V.33. *Expand* $(a+b)^7$

Definition V.33.1: Coefficients of a binomial expansion

Let n and r be nonnegative integers with $n \geq r$. The expression $\binom{n}{r}$ (read as "n choose r") is called a binomial coefficient and is defined by

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Other notations for $\binom{n}{r}$ are ${}_nC_r$ and $C(n,r)$

EXAMPLE V.34. *Evaluate*

A $\binom{4}{2}$

B $\binom{10}{7}$

C $\binom{10}{3}$

D $\binom{5}{0}$

E $\binom{5}{5}$

Example V.35. *Evaluate*

A $\binom{n}{0}$ **B** $\binom{n}{n}$

D $\binom{n}{1}$ **E** $\binom{n}{n-1}$

Remark. $\binom{n}{r} = \binom{n}{n-r} = \frac{n!}{r!(n-r)!}$

Objective 68: Apply the binomial theorem

The binomial theorem can be proved with mathematical induction.

Theorem V.35.1: The Binomial Theorem

Let n be a positive integer. The expansion of $(a+b)^n$ is given by

$$\begin{aligned}(a+b)^n &= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n \\ &= \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r\end{aligned}$$

Example V.36. *Expand using the binomial theorem* $(3y+2z)^5$

Example V.37. *Expand using the binomial theorem* $(y^2-5z)^4$

Objective 69: Find a specific term in a binomial expansion

Sometimes, only a particular coefficient is needed, instead of all the terms in a complete expansion

Specific Term of a Binomial Expansion

The form of each term in the expansion of $(a+b)^n$ is

$$\binom{n}{r} a^{n-r} b^r$$

where $r = 0, 1, 2, ..., n$. To find the kth term, let $r = k - 1$

EXAMPLE V.38. *Find the 7th term of the expansion of* $(x - 3y)^9$

Students - Check your understanding.

CYU 24. *Expand the expression* $\left(\sqrt{x} + \sqrt{5}\right)^6$ *using the Binomial Theorem.*