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Preface

Notes for MTG 5326.0001 Topology I taught at FSU in Fall 2018 by Dr. Sam Ballas.

In these notes, the notation $A \subset X$ means that A is a subset of X, which may also refer to being an improper subset and 2^X refers to the powerset of X, which we assume to always exist for any X. Although $\varnothing$ is personally preferred to represent the empty set but in these notes, the symbol $\emptyset$ will take precedence. The complement of A in X will be denoted by A^c, $X - A$ or even $X \backslash A$. Problems which were assigned for homework are scattered in (hopefully) corresponding sections, sometimes labelled as such while other times, simply labelled as propositions.

Syllabus

Text:	Topology, James Munkres, 2nd Edition optional
Eligibility:	Graduate standing or permission of the instructor.
Course Content:	This course will cover basic concepts in point set topology. It will loosely follow Munkres book
Grading:	The grade distribution for this course is as follows:

Homework	40%
Midterms	30%
Final Exam	30%

The following numerical grade will guarantee you at least the corresponding letter grade, although depending on the performance of the class the grade cutoffs may be lower:

A: 90-100; B: 80-89; C: 70-79; D: 60-69; F: 0-59.

Plus or minus grades may be assigned. A grade of I (incomplete) will not be given to avoid a grade of F or to give additional study time. Failure to process a course drop will result in a course grade of F.

Exams: There will be a 2 midterm exams and a cumulative final. These exams will be taken during class. The exam schedule is as follows:

Exam 1:	Thursday, Oct 4
Exam 2:	Thursday, Nov 15
Final:	Wednesday, Dec 12, 7:30-9:30a.

Homework: In my experience the best way to learn this type of mathematics is by getting your hands dirty with problems of varying degrees of difficulty. For this reason I place a particular emphasis on homework in this course. There will be several homework assignments due throughout the semester. I encourage you to start these assignments well in advance of their due dates and expect you to spend a good amount of time on each one. I also strongly encourage you to discuss the problems the problems with your classmates, however, the work you submit must be your own.

The assignments will be posted on the Canvas page for our course. If you are enrolled in the course and having difficulty accessing the Canvas page please contact me immediately so that we can resolve the issue and you don't end up missing any homework assignments.

You will be required to TEXthe solutions to your homework assignments. If you need help getting TEXup and running on your computer please let me know. I will post a template file for the homework to give you an idea what I want your solutions to look like. You are not required to use the template, but should roughly follow its structure when preparing your homework solutions. You will submit a pdf version of your homework via Canvas. When submitting I would appreciate if your would follow the naming convention homework#N_yourLastName.pdf

At the end of the semester, I will automatically drop your lowest homework grade.

Expectations: I expect that everyone will maintain a classroom conducive to learning. I like an informal atmosphere, but it must be orderly. Thus, everyone is expected to behave with basic politeness, civility, and respect for others. In particular, talking in class is OK if it's part of a class discussion or with me. Private communications are not permitted, especially during quizzes and tests. I also expect that when you are in class that the mathematics at hand will receive your undivided attention. Indicators that your attention is divided include, but are not limited to:

- Texting,
- Using social media (Facebook/Twitter/etc.), or
- Playing games on your cell phone.

University Attendance Policy: Excused absences include documented illness, deaths in the family and other documented crises, call to active military duty or jury duty, religious holy days, and official University activities. These absences will be accommodated in a way that does not arbitrarily penalize students who have a valid excuse. Consideration will also be given to students whose dependent children experience serious illness.

Academic Honor Policy: The Florida State University Academic Honor Policy outlines the University's expectations for the integrity of students' academic work, the procedures for resolving alleged violations of those expectations, and the rights and responsibilities of students and faculty members throughout the process. Students are responsible for reading the Academic Honor Policy and for living up to their pledge to "...be honest and truthful and ... [to] strive for personal and institutional integrity at Florida State University." (Florida State University Academic Honor Policy, found at http://fda.fsu.edu/Academics/AcademicHonorPolicy.) To summarize, violations of these policies will result in a rather messy affair for you and me, so just don't do it.

American's with Disabilities Act: Students with disabilities needing academic accommodation should: (1) register with and provide documentation to the Student Disability Resource Center; and (2) bring a letter to the instructor indicating the need for accommodation and what type. Please note that instructors are not allowed to provide classroom accommodation to a student until appropriate verification from the Student Disability Resource Center has been provided.

This syllabus and other class materials are available in alternative format upon request. For more information about services available to FSU students with disabilities, contact the Student Disability Resource Center

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Syllabus Change Policy: Except for changes that substantially affect implementation of the evaluation (grading) statement, this syllabus is a guide for the course and is subject to change with advance notice.

1 Metric Spaces

Definition 1 *Let X be any non-empty set. A **metric** defined on X is a function $d : X \times X \longrightarrow [0, \infty)$ such that*

- *M1* $d(x, y) \geq 0$
- *M2* $d(x, y) = 0 \iff x = y$
- *M3* $d(x, y) = d(y, x)$
- *M4* $d(x, y) \leq d(x, z) + d(z, y)$

for $x, y, z \in X$. A **metric space** is a pair (X, d) where X is a set and d is a metric on X. Metric generalises the concept of distance between two points.

Example 2 *A trivial metric which can be defined on any set is the **discrete metric**, which is defined as $d(x, y) = 1$ for $x \neq y$ and 0 otherwise. Clearly, $d(x, y) \geq 0, d(x, y) = 0 \iff x = y$ and $d(x, y) = d(y, x)$ are satisfied by definition. $d(x, y) \leq d(x, z) + d(z, y)$ can be verified exhaustively by considering cases $x \neq y$, $x = y$, $x \neq z$ and $x = z$.*

Example 3 *On the real line $\mathbb{R}$, we can define the **usual metric** $d(x, y) = |x - y|$. This can be generalised for the Euclidean plane $\mathbb{R}^n$ with metric*

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2}$$

For $n = 1$, we have $\sqrt{(x_1 - y_1)^2} = |x - y|$. For $n = 2$, we have the familiar Pythagorean Theorem in a plane. This metric is the generalized Pythagorean Theorem in $\mathbb{R}^n$.

The above can be generalised as follows:

Problem 4 *Let $p > 0$ and define $d_p : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $d_p ((x_1, ..., x_n), (y_1, ..., y_n)) = (|x_1 - y_1|^p + ... + |x_n - y_n|^p)^{1/p}$*

1. *Show that if $p \geq 1$ then d_p gives a metric on $\mathbb{R}^n$. This metric is called the L^P metric*
2. *Show that if $0 < p < 1$ then d_p does not give a metric on $\mathbb{R}^n$*

Solution 5 *1. Let $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$. Let $i \in \{1, ..., n\}$. For **M1**, since $x_i, y_i \in \mathbb{R} \Longrightarrow x_i - y_i \in \mathbb{R}$ by closure of addition $\Longrightarrow |x_i - y_i| \in \mathbb{R}^+$ because valuation on a field is positive-valued. Hence for $p = \infty$, $d_\infty (x, y) = \max\limits_{1 \leq i \leq n} \{|x_i - y_i|\} \in \mathbb{R}^+$. Now, let $1 \leq p < \infty$. Then, $|x_i - y_i| \in \mathbb{R}^+ \Longrightarrow |x_i - y_i|^p \in \mathbb{R}^+$ by closure of multiplication. By closure of addition, $\sum |x_i - y_i|^p \in \mathbb{R}^+$. Again, since $\mathbb{R}$ is a (ordered) field, for all $a \in \mathbb{R}^+$, there exists a b such* *that* $b^p = a$. *Letting* $a = \sum_{i=1}^{n} |x_i - y_i|^p$ *and* $b = d_p(\mathbf{x}, \mathbf{y})$ *shows that* $d_p(\mathbf{x}, \mathbf{y})$ *is non-negative and real-valued.* ***M2***, *for* $1 \leq p < \infty$

$$\begin{aligned} d_p(\mathbf{x}, \mathbf{y}) &= 0 \\ &\iff \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p} = 0 \\ &\iff \sum_{i=1}^{n} |x_i - y_i|^p = 0 \\ &\iff |x_i - y_i|^p = 0 \text{ for all } i \\ &\iff |x_i - y_i| = 0 \ \forall p \geq 1 \\ &\iff x_i = y_i \\ &\iff \mathbf{x} = \mathbf{y} \end{aligned}$$

For $p = \infty$, $d_\infty(\mathbf{x}, \mathbf{y}) = \max_{1 \leq i \leq n} \{|x_i - y_i|\} = 0$ *implies* $|x_i - y_i| = 0$ *for all* i. *Hence* $x_i = y_i$ *for all* $i \implies \mathbf{x} = \mathbf{y}$. *Next, for* ***M3***,

$$d_p(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p} = \left(\sum_{i=1}^{n} |y_i - x_i|^p \right)^{1/p} = d_p(\mathbf{y}, \mathbf{x})$$

Similarly, For $p = \infty$, $d_\infty(\mathbf{x}, \mathbf{y}) = \max_{1 \leq i \leq n} \{|x_i - y_i|\} = \max_{1 \leq i \leq n} \{|y_i - x_i|\} = d_\infty(\mathbf{y}, \mathbf{x})$. *Finally, for* ***M4***, *let* $\mathbf{z} = (z_1, ..., z_n)$. *We start with* $p = 1$. *In this case, we can directly apply the triangle inequality to get the desired result.*

$$\begin{aligned} d_1(\mathbf{x}, \mathbf{y}) &= \sum_{i=1}^{n} |x_i - y_i| \\ &= \sum_{i=1}^{n} |x_i - z_i + z_i - y_i| \\ &\leq \sum_{i=1}^{n} |x_i - z_i| + \sum_{i=1}^{n} |z_i - y_i| \\ &= d_1(\mathbf{x}, \mathbf{z}) + d_1(\mathbf{z}, \mathbf{y}) \end{aligned}$$

For $p > 1$, *we follow the proof found in Kreyzig's Functional Analysis for Minkowski's inequality. Let* $1 \leq i \leq n$. *We first let* q *be a natural number such that* $\frac{1}{p} + \frac{1}{q} = 1$ *from which we have* $\frac{q+p}{pq} = 1$ *and therefore* $pq - p - q = 0 \implies pq - p - q + 1 = 1 \implies p(q-1) - (q-1) = 1 \implies (p-1)(q-1) = 1$. *Thus,*

$$\frac{1}{p-1} = q - 1$$

Now, let $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be a function defined by* $f(t) = u = t^{p-1}$. *We can have* $u^{1/(p-1)} = t$ *or* $f^{-1}(u) = t = u^{q-1}$. *Now let* $a, b \in \mathbb{R}$ *for* $a, b > 0$. *If we think of* *ab as an area of a rectangle with sides a and b, then*

$$\begin{aligned} ab &\leq \int_0^a f(t)\,dt + \int_0^b f^{-1}(t)\,dt \\ &= \int_0^a t^{p-1}dt + \int_0^b t^{q-1}dt \\ &= \frac{a^p}{p} + \frac{b^q}{q} \qquad (1) \end{aligned}$$

Now let $\xi_i, y_i \in \mathbb{R}$ *such that*

$$\sum_{i=1}^{n} |\xi_i|^p = \sum_{i=1}^{n} |\eta_i|^q = 1$$

Set $a = |\xi_i|$ *and* $b = |\eta_i|$ *and plug values in* (1)*. Then, we have the inequality*

$$|\xi_i|\,|\eta_i| \leq \frac{|\xi_i|^p}{p} + \frac{|\eta_i|^q}{q}$$

Summing over i, we get the inequality

$$\sum_{i=1}^{n} |\xi_i|\,|\eta_i| \leq \sum_{i=1}^{n} \frac{|\xi_i|^p}{p} + \sum_{i=1}^{n} \frac{|\eta_i|^q}{q}$$

from which we have

$$\sum_{i=1}^{n} |\xi_i|\,|\eta_i| = \sum_{i=1}^{n} |\xi_i \eta_i| \leq \frac{1}{p} + \frac{1}{q} = 1 \qquad (2)$$

Now, let

$$\xi_i = \frac{x_i}{\left(\Sigma\,|x_i|^p\right)^{1/p}}$$

and

$$\eta_i = \frac{y_i}{\left(\Sigma\,|y_i|^q\right)^{1/q}}$$

Then,

$$|\xi_i|^p = \frac{|x_i|^p}{\left(\Sigma\,|x_i|^p\right)}$$

and

$$|\eta_i|^q = \frac{|y_i|^q}{\left(\Sigma\,|y_i|^q\right)}$$

which, on summing over the index i , will yield

$$\sum_{i=1}^{n} |\xi_i|^p = \frac{\Sigma\,|x_i|^p}{\left(\Sigma\,|x_i|^p\right)} \;\; \textit{and} \; \sum_{i=1}^{n} |\eta_i|^q = \frac{\Sigma\,|y_i|^q}{\left(\Sigma\,|y_i|^q\right)}$$

both of which equal 1. Hence,

$$\xi_i = \frac{x_i}{\left(\Sigma \left|x_i\right|^p\right)^{1/p}} \text{ and } \eta_i = \frac{y_i}{\left(\Sigma \left|y_i\right|^q\right)^{1/q}}$$

is a valid substitution in (2). *We then get*

$$\begin{aligned} 1 \quad &\geq \quad \sum_{i=1}^{n} \left| \frac{x_i}{\left(\Sigma \left|x_i\right|^p\right)^{1/p}} \frac{y_i}{\left(\Sigma \left|y_i\right|^q\right)^{1/q}} \right| \\ &\implies \quad \sum_{i=1}^{n} \left|x_i y_i\right| \leq \left(\sum_{i=1}^{n} \left|x_i\right|^p\right)^{1/p} \left(\sum_{i=1}^{n} \left|y_i\right|^q\right)^{1/q} \end{aligned}$$

which is the famous Hölder inequality. Now, from the triangle inequality, we note that

$$\left|x_i + y_i\right|^p = \left|x_i + y_i\right| \left|x_i + y_i\right|^{p-1} \leq \left(\left|x_i\right| + \left|y_i\right|\right) \left|x_i + y_i\right|^{p-1}$$

Summing over i*, we get*

$$\sum_{i=1}^{n} \left|x_i + y_i\right|^p \leq \sum_{i=1}^{n} \left|x_i\right| \left|x_i + y_i\right|^{p-1} + \sum_{i=1}^{n} \left|y_i\right| \left|x_i + y_i\right|^{p-1} \tag{3}$$

The first term on the right side of the inequality, after applying Holder's inequality, becomes

$$\sum_{i=1}^{n} \left|x_i\right| \left|x_i + y_i\right|^{p-1} \leq \left(\sum_{i=1}^{n} \left|x_i\right|^p\right)^{1/p} \left(\sum_{i=1}^{n} \left|x_i + y_i\right|^{(p-1)q}\right)^{1/q}$$

Since $pq = p + q$*, we must have* $(p-1)\, q = pq - q = p$*. Hence*

$$\begin{aligned} &\sum_{i=1}^{n} \left|x_i\right| \left|x_i + y_i\right|^{p-1} \leq \left(\sum_{i=1}^{n} \left|x_i\right|^p\right)^{1/p} \left(\sum_{i=1}^{n} \left|x_i + y_i\right|^{(p-1)q}\right)^{1/q} \\ = \quad &\left(\sum_{i=1}^{n} \left|x_i\right|^p\right)^{p-1} \left(\sum_{i=1}^{n} \left|x_i + y_i\right|^p\right)^{1/q} \end{aligned}$$

Similarly, the second term of (3) *becomes*

$$\sum_{i=1}^{n} \left|y_i\right| \left|x_i + y_i\right|^{p-1} \leq \left(\sum_{i=1}^{n} \left|y_i\right|^p\right)^{1/p} \left(\sum_{i=1}^{n} \left|x_i + y_i\right|^p\right)^{1/q}$$

Hence we have

$$\begin{aligned} \sum_{i=1}^{n} \left|x_i + y_i\right|^p \quad &\leq \quad \left(\left(\sum_{i=1}^{n} \left|x_i\right|^p\right)^{p-1} \left(\sum_{i=1}^{n} \left|x_i + y_i\right|^p\right)^{1/q}\right) + \left(\left(\sum_{i=1}^{n} \left|y_i\right|^p\right)^{1/p} \left(\sum_{i=1}^{n} \left|x_i + y_i\right|^p\right)^{1/q}\right) \\ &= \quad \left(\left(\sum_{i=1}^{n} \left|x_i\right|^p\right)^{1/p} + \left(\sum_{i=1}^{n} \left|y_i\right|^p\right)^{1/p}\right) \left(\sum_{i=1}^{n} \left|x_i + y_i\right|^p\right)^{1/q} \end{aligned}$$

$$\Longrightarrow \left(\sum_{i=1}^{n}|x_i+y_i|^p\right)^{p-1/q} \leq \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n}|y_i|^p\right)^{1/p}$$

Since $\frac{1}{p}+\frac{1}{q}=1$, we must have $1-\frac{1}{q}=\frac{1}{p}$. Hence the above inequality becomes

$$\left(\sum_{i=1}^{n}|x_i+y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n}|y_i|^p\right)^{1/p}$$

Now, replace x_i with $x_i - z_i$ and replace y_i with $z_i - y_i$ to get

$$\begin{aligned}
\left(\sum_{i=1}^{n}|x_i-z_i+z_i-y_i|^p\right)^{1/p} &= \left(\sum_{i=1}^{n}|x_i-y_i|^p\right)^{1/p} = d_p(\mathbf{x},\mathbf{y}) \\
&\leq \left(\sum_{i=1}^{n}|x_i-z_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n}|z_i-y_i|^p\right)^{1/p} \\
&= d_p(\mathbf{x},\mathbf{z}) + d_p(\mathbf{z},\mathbf{y})
\end{aligned}$$

and ***M4*** *is satisfied for $1<p<\infty$. For the case of $p=\infty$, we have*

$$\begin{aligned}
|x_i-y_i| &\leq |x_i-z_i|+|z_i-y_i| \\
&\leq |x_i-z_i| + \max_{1\leq i\leq n}|z_i-y_i| \\
&\leq \max_{1\leq i\leq n}|x_i-z_i| + \max_{1\leq i\leq n}|z_i-y_i|
\end{aligned}$$

That is,

$$|x_i-y_i| \leq \max_{1\leq i\leq n}|x_i-z_i| + \max_{1\leq i\leq n}|z_i-y_i| \text{ for all } i$$

$$\Longrightarrow \max_{1\leq i\leq n}|x_i-y_i| \leq \max_{1\leq i\leq n}|x_i-z_i| + \max_{1\leq i\leq n}|z_i-y_i|$$

or that $d_\infty(\mathbf{x},\mathbf{y}) \leq d_\infty(\mathbf{x},\mathbf{z}) + d_\infty(\mathbf{z},\mathbf{y})$

2. Let $x=(1,1,0,0,...,0)$, $y=(1,0,...,0)$ and $z=(0,....,0)$ and $p\in(0,1)$. Then, $d_p(\mathbf{x},\mathbf{z}) = \left(\sum_{i=1}^{n}|x_i-z_i|^p\right)^{1/p} = 2^{1/p} > 2$, $d_p(\mathbf{y},\mathbf{z})=1$ and $d_p(\mathbf{x},\mathbf{y})=1$ and so $d_p(\mathbf{x},\mathbf{z}) \nleq d_p(\mathbf{x},\mathbf{y})+d_p(\mathbf{y},\mathbf{z})$. That is, the triangle inequality fails.

Example 6 *The case for $p=1$ is the metric.*

$$d(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n}|x_i-y_i|$$

This is called the ***taxicab*** *or* ***Manhattan metric****.*

This example illustrates the important fact that from a given set with more than one element, we can obtain various metric spaces by choosing different metric functions.

Example 7 *A metric similar to the usual metric can also be defined for the complex plane* $\mathbb{C}$ *by using the definition of the modulus* $|z| = \sqrt{x^2+y^2}$ *for* $z = x+iy$ *and have a metric. Thus,* $d(z,w) = |z-w| = \sqrt{(x-u)^2+(y-v)^2}$ *where* $w = u+iv$*. Thus, we can now easily generalize this definition for* $\mathbb{C}^n$ *by* $z = (z_1, ..., z_n)$*,* $w = (w_1, ..., w_n)$ *and*

$$d(z,w) = \sqrt{(x_1-u_1)^2+(y_1-v_1)^2+...+(x_n-u_n)^2+(y_n-v_n)^2}$$

Example 8 *As a set* l^∞*, we take the set of all bounded sequences of complex (or real) numbers; that is, every element* x *of* l^∞ *is a complex (resp. real) sequence* $x = (\xi_1, \xi_2, ...)$*, briefly* $x = (\xi_i)$*. If we have* $x = (\xi_i)$ *and* $y = (\varsigma_i)$*, we can have the metric defined by* $d(x,y) = \sup A$ *where* $A = \{\alpha_i \mid \alpha_i = |\xi_i - \varsigma_i|\}$*. This can be compactly written as*

$$\sup_{i\in\mathbb{N}} |\xi_i - \varsigma_i|$$

The supremum exists since the set is bounded and is unique. To prove that this is a metric is easy: ***M1****,* ***M2*** *and* ***M3*** *can be easily satisfied. For* ***M4****, let* $x = (\xi_i)$*,* $y = (\varsigma_i)$ *and* $z = (\eta_i)$*. Then,* $|\xi_i - \varsigma_i| = |\xi_i - \eta_i + \eta_i - \varsigma_i| \le |\xi_i - \eta_i| + |\eta_i - \varsigma_i|$ $\forall i$

$$\begin{aligned}
&\Longrightarrow |\xi_i - \varsigma_i| \le \sup_{i\in\mathbb{N}} |\xi_i - \eta_i| + |\eta_i - \varsigma_i| \ \forall i \\
&\Longrightarrow |\xi_i - \varsigma_i| \le \sup_{i\in\mathbb{N}} |\xi_i - \eta_i| + \sup_{i\in\mathbb{N}} |\eta_i - \varsigma_i| \ \forall i \\
&\Longrightarrow \sup_{i\in\mathbb{N}} |\xi_i - \varsigma_i| \le \sup_{i\in\mathbb{N}} |\xi_i - \eta_i| + \sup_{i\in\mathbb{N}} |\eta_i - \varsigma_i| \\
&\Longrightarrow d(x,y) \le d(y,z) + d(z,x).
\end{aligned}$$

Example 9 *For* $C([a,b], \mathbb{R})$*, usually abbreviated as* $C[a,b]$*, the set of real-valued functions from* $[a,b]$*, we have a bona fide metric space under the metric* $d(x,y) = \max_{t\in[a,b]} |x(t) - y(t)|$*.*

Example 10 *Let* S *be a non-empty set and let* $B(S)$ *be the space of bounded functions on* S*. Define the metric*

$$d(x,y) = \sup_{t\in s} |x(t) - y(t)|$$

Clearly, $d(x,y) \ge 0$*.* ***M2****,*

$$\begin{aligned}
d(x,y) &= 0 \\
&\Longleftrightarrow \sup_{t\in S} |x(t) - y(t)| = 0 \\
&\Longleftrightarrow |x(t) - y(t)| = 0 \forall t \in S
\end{aligned}$$

since if the supremum of non-negative numbers zero, then all the numbers are *themselves zero. Then, we have* $x(t) = y(t)\ \forall t \in S$. *Hence,* $x = y$. ***M3***,

$$\begin{aligned} d(x,y) &= \sup_{t\in S} |x(t) - y(t)| \\ &= \sup_{t\in S} |-(x(t) - y(t))| \\ &= \sup_{t\in S} |-x(t) + y(t)| \\ &= \sup_{t\in S} |y(t) - x(t)| \\ &= d(y,x) \end{aligned}$$

Finally,

$$\begin{aligned} |x(t) - y(t)| &\leq |x(t) - z(t)| + |z(t) - y(t)| \text{ for all } t \in S \\ &\leq |x(t) - z(t)| + \sup_{t\in S} |z(t) - y(t)| \\ &\leq \sup_{t\in S} |x(t) - z(t)| + \sup_{t\in S} |z(t) - y(t)| \end{aligned}$$

Thus, $\sup_{t\in S} |x(t) - y(t)| \leq \sup_{t\in S} |x(t) - z(t)| + \sup_{t\in S} |z(t) - y(t)|$.

With the next example, we will explore how to define new metrics from old ones.

Example 11 *Let* $d(x,y)$ *be a metric. Define* $\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}$. *Then,* ρ *is a metric. In fact, let's have fun and go infinite dimensional. We can have a metric on the space* s *of all bounded and unbounded sequences defined as*

$$d(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \varsigma_i|}{1 + |\xi_i - \varsigma_i|}$$

for $x = (\xi_i)$, $y = (\varsigma_i) \in \mathbb{R}^{\infty}$. *Note that* $|\xi_i - \varsigma_i| = d_i(\xi_i - \varsigma_i)$ *is a metric for all* i.

Proving ***M1*** *to* ***M3*** *is easy. For* ***M4***, *let* $z = (\eta_i)$ *such that*

$$\begin{aligned} |\xi_i - \varsigma_i| &\leq |\xi_i - \eta_i| + |\eta_i - \varsigma_i| \\ &\Longrightarrow \frac{1}{1 + |\xi_i - \eta_i|} + \frac{1}{1 + |\eta_i - \varsigma_i|.} \\ &\leq \frac{1}{2 + |\xi_i - \varsigma_i|} \\ &\leq \frac{1}{1 + |\xi_i - \varsigma_i|} \end{aligned}$$

Note that

$$\begin{aligned} d(x,y) &= \sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|\xi_i-\varsigma_i|}{1+|\xi_i-\varsigma_i|} \\ &= \sum_{i=1}^{\infty}\frac{1}{2^i}\frac{-1+1+|\xi_i-\varsigma_i|}{1+|\xi_i-\varsigma_i|} \\ &= \sum_{i=1}^{\infty}\frac{1}{2^i}\left(1-\frac{1}{1+|\xi_i-\varsigma_i|}\right) \\ &= \sum_{i=1}^{\infty}\frac{1}{2^i}-\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{1}{1+|\xi_i-\varsigma_i|} \\ &= 1-\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{1}{1+|\xi_i-\varsigma_i|} \end{aligned}$$

Since we have $\frac{1}{2^i}\frac{1}{1+|\xi_i-\eta_i|}+\frac{1}{2^i}\frac{1}{1+|\eta_i-\varsigma_i|.}\leq\frac{1}{2^i}\frac{1}{1+|\xi_i-\varsigma_i|}$, *we can equivalently have*

$$\begin{aligned} & 1-\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{1}{1+|\xi_i-\eta_i|}+1-\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{1}{1+|\eta_i-\varsigma_i|.} \\ \geq\; & 1-\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{1}{1+|\xi_i-\varsigma_i|} \end{aligned}$$

which is what is required.

Problem 12 *Let* (X,d) *be a metric space. Define* $\overline{d}:X\times X\longrightarrow\mathbb{R}$ *by* $\overline{d}(x,y)=\min(d(x,y),1)$. *Prove that* $\overline{d}$ *is also a metric on* X.

Such a metric $\overline{d}$ is called the **standard bounded metric** associated with d.

Solution 13 ***M1*** *Since* $d(x,y)\in\mathbb{R}^+$ *for all* $x,y\in X$ *therefore* $\overline{d}(x,y)=\min(d(x,y),1)\in\mathbb{R}^+$

M2 $\overline{d}(x,y)=0\iff\min(d(x,y),1)=0\iff d(x,y)=0\iff x=y$

M3 $\overline{d}(x,y)=\min(d(x,y),1)=\min(d(y,x),1)=\overline{d}(y,x)$

M4 *First, we note that , since d is a metric, it follows that* $d(x,y)\leq d(x,z)+d(z,y)$ *for all* $x,y,z\in X$. *Also, we note that either* $\min(d(x,y),1)=d(x,y)$ *or* 1. *For the former, we must have* $d(x,y)\leq 1$ *for all* $x,y\implies\min(d(x,y),1)\leq 1$. *If* $\min(d(x,y),1)=1$, *then again* $\min(d(x,y),1)\leq 1$ *trivially. Hence* $\overline{d}(x,y)\leq 1$ *for all* x,y *in any case. Now, we can reduce the problem to two cases.*

Case I

$\min(d(x,z),1)=1$ *or* $\min(d(z,y),1)=1$. *Then,* $\overline{d}(x,y)\leq 1\leq 1+1=\min(d(x,z),1)+\min(d(z,y),1)=\overline{d}(x,z)+\overline{d}(z,y)$

Case II

$\min(d(x,z),1) \neq 1$ *and* $\min(d(z,y),1) \neq 1$. *It cannot be that* $\min(d(x,z),1) > 1$ *and* $\min(d(z,y),1) > 1$. *Thus, we must have* $\min(d(x,z),1) < 1$ *and* $\min(d(z,y),1) < 1$. *Then,* $\min(d(x,z),1) + \min(d(z,y),1) = d(x,z) + d(z,y) \geq d(x,y) \geq \min(d(x,y),1)$. *That is,* $\overline{d}(x,z) + \overline{d}(z,y) = d(x,z) + d(z,y) \geq d(x,y) \geq \overline{d}(x,y)$

In either case, $\overline{d}(x,y) \leq \overline{d}(x,z) + \overline{d}(z,y)$ *for all* $x,y,z \in X$.

In general, we have the following:

Proposition 14 *Let d be a metric and let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a function such that $f(0) = 0$, $f(z+w) \leq f(z) + f(w)$ (in this case, f is said to be sub-additive) and $x \leq y \Longrightarrow f(x) \leq f(y)$ (that is, f is non-decreasing). Then, $\rho(x,y) = f(d(x,y))$ is a new metric.*

Proof. For $0 \leq d(x,y)$, we have $0 = f(0) \leq f(d(x,y)) = \rho(x,y)$ and so M1 holds. For M2, by $f(0) = 0$, we have $0 = f(d(x,y)) = \rho(x,y) \iff d(x,y) = 0 \iff x = y$. For M3, $\rho(x,y) = f(d(x,y)) = f(d(y,x)) = \rho(y,x)$. Finally, $d(x,y) \leq d(x,z) + d(z,y) \Longrightarrow f(d(x,y)) \leq f(d(x,z) + f(d(z,y)))$ by non-decreasing property of f and that $f(d(x,z) + f(d(z,y))) \leq f(d(x,z)) + f(d(z,y))$ by sub-additivity of f from which we have $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$ for all x,y,z. ■

Example 15 $f(x) = \frac{x}{1+x}$ *is the "toned-down" example of the above infinite-dimensional case. Clearly, $f(0) = 0$. Also, $f'(x) = \frac{1}{(1+x)^2} > 0$ so that f is non-decreasing. Finally,* $f(z+w) = \frac{z}{1+z+w} + \frac{w}{1+z+w} \leq \frac{z}{1+z} + \frac{w}{1+w} = f(z) + f(w)$.

Example 16 *Another example is $f(x) = \lceil x \rceil$, the ceil function. In this case, $f(0) = 0$, $x \leq y \Longrightarrow \lceil x \rceil \leq \lceil y \rceil$ and $\lceil x+y \rceil \leq \lceil x \rceil + \lceil y \rceil$.*

Here's a different way:

Problem 17 *Let (X_i, d_i) be metric spaces for $1 \leq i \leq n$. Let $Y = X_1 \times ... \times X_n$ and define $d : Y \times Y \longrightarrow \mathbb{R}$ by $d(\mathbf{x}, \mathbf{y}) = \max\{d_i(x_i, y_i)\}$ for $1 \leq i \leq n$ and $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$ in Y. Show that d is a metric on Y*

Solution 18 *For non-negativity **M1**: d_i is a metric for all i implies $d_i(x_i, y_i) \geq 0$ for all $i \Longrightarrow \max\{d_i(x_i, y_i)\} \geq 0 \Longrightarrow d(\mathbf{x}, \mathbf{y}) \geq 0$. For non-degeneracy **M2**, $d(\mathbf{x}, \mathbf{y}) = 0 \iff \max\{d_i(x_i, y_i)\} = 0$. Since maximum is taken over all non-negative values, then $\max\{d_i(x_i, y_i)\} = 0 \iff d_i(x_i, y_i) = 0$ for all i. Hence $x_i = y_i$ for all $i \iff (x_1, x_2, ..., x_n) = (y_1, y_2, ..., y_n) \iff \mathbf{x} = \mathbf{y}$. Symmetry **M3**: $d(\mathbf{x}, \mathbf{y}) = \max\{d_i(x_i, y_i)\} = \max\{d_i(y_i, x_i)\} = d(\mathbf{y}, \mathbf{x})$. Triangle inequality **M4**: let $1 \leq i \leq n$ and $\mathbf{z} = (z_1, z_2, ..., z_n) \in Y$. Then, since $d_i(x_i, y_i)$ is a metric for each i, it follows that, for all i*

$$\begin{aligned} d_i(x_i, y_i) &\leq d_i(x_i, z_i) + d_i(z_i, y_i) \\ &\leq d_i(x_i, z_i) + \max_i\{d_i(z_i, y_i)\} \\ &\leq \max_i\{d_i(x_i, z_i)\} + \max_i\{d_i(z_i, y_i)\} \end{aligned}$$

That is, $d_i(x_i, y_i) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$. *Since this is true for all* i*, it must be true that*

$$\max_i \{d_i(x_i, y_i)\} \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$$

In other words, $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$

Problem 19 *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces. A map* $f : X \longrightarrow Y$ *is called an isometric embedding if* $d_Y(f(x), f(y)) = d_X(x, y)$ *for all* $x, y \in X$*. Prove that an isometric embedding is always injective.*

Solution 20 *Let* $f(x) = f(y)$. *Then,* $0 = d_Y(f(x), f(y)) = d_X(x, y) = 0 \implies x = y$.

1.1 Balls and Spheres

In a metric space (X, d), we have the following

Definition 21 *Given a point* $x_0 \in X$ *and a real number* $r > 0$*, we define three types of sets:*

1. $N_r(x_0) = B(x_0; r) = \{x \in X \mid d(x, x_0) < r\}$ (Open ball)
2. $\overline{N_r(x_0)} = \overline{B(x_0; r)} = \{x \in X \mid d(x, x_0) \leq r\}$ (Closed ball)
3. $S(x_0; r) = \{x \in X \mid d(x, x_0) = r\}$ (Sphere)

Intuitively, it is clear that in all three cases, x_0 is the centre and r the radius. Mathematically, the open ball of radius r is the set of all points in X whose distance from the centre of the ball is than r.

The open ball with an ϵ-radius is called an open neighborhood. We can have this when we replace r with ϵ in an open ball. Thus, $N_\epsilon(x_0)$ is an **ϵ-neighbourhood** of x_0 where $\epsilon > 0$. This ϵ can be arbitrary (but positive!). Trivially, every neighbourhood of x_0 contains x_0 so that $N_\epsilon(x_0) \neq \emptyset$.

Definition 22 *Let* (X, d) *be a metric space and* $U \subset X$. $x \in X$ *is said to be an* ***interior point*** *of* U *if there exists* $\epsilon > 0$ *such that* $N_\epsilon(x) \subset U$. U *is said to be* ***open*** *if it contains an open neighborhood about each of its points. That is,* $\forall x \in U$*, there exists* $\epsilon > 0$ *such that* $N_\epsilon(x) \subset U$*. That is, every point is an interior point.* $K \subset X$ *is said to be* ***closed*** *if* $K^c = X - K = X \backslash K$ *(same notation) is open.*

With this in hand, let us see a very strange topology.

Problem 23 *Let* $p \in \mathbb{Z}$ *be a prime number, let* $x \in \mathbb{Q}$ *and write* $x = \frac{a}{b}p^c$ *where* $a, b, c \in \mathbb{Z}$ *are integers and* a *and* b *are relatively prime integers. Define* $|x|_p = p^{-c}$ *and* $d_p : \mathbb{Q} \times \mathbb{Q} \longrightarrow \mathbb{R}$ *by* $d_p(x, y) = |x - y|_p$. d_p *is called the* p*-adic metric on* $\mathbb{Q}$.

1. *Show that d_p satisfies the strong triangle inequality, namely that $d_p(x,y) \leq \max\{d_p(x,z), d_p(z,y)\}$*

2. *Show that d_p is a metric (this metric is called the p-adic metric)*

3. *Show that all triangles in $(\mathbb{Q}, d_p)$ are isosceles. That is, if $x,y,z \in \mathbb{Q}$ then at least one of the equalities $d_p(x,z) = d_p(x,y)$, $d_p(y,z) = d_p(x,y)$ or $d_p(x,z) = d_p(y,z)$ is satisfied.*

4. *Let $x \in \mathbb{Q}$, $r > 0$ a real number, let $N_r(x)$ be the r-neighborhood around x. We have seen that r-neighborhoods are always open in any metric space. Show that if $C = (N_r(x))^c$ then C is also open. In other words, open balls are also closed.*

Solution 24 ***1.*** *Let $x,y,z \in \mathbb{Q}$. Without loss of generality, assume that $|x-z|_p \leq |z-y|_p$. Since x,y,z are rational numbers and their difference is another rational number, we can write $x - z = \frac{a}{b}p^c$ and $z - y = \frac{a'}{b'}p^{c'}$ such that $\gcd(a,b) = \gcd(a',b') = 1$. Then, $|x-z|_p = p^{-c}$ and $|z-y|_p = p^{-c'}$. By choice of x,y,z, we must have $p^{-c} \leq p^{-c\prime} \implies p^{c'} \leq p^c \implies c' \leq c$. Now, $d(x,y) = |x-y|_p = |x-z+z-y|_p = \left|\frac{a}{b}p^c - \frac{a'}{b'}p^{c'}\right|_p = \left|p^{c'}\left(\frac{a}{b}p^{c-c'} - \frac{a'}{b'}\right)\right|_p$. We can write $\frac{a}{b}p^{c-c'} - \frac{a'}{b'}$, a rational number, as $\frac{ab'p^{c-c'} - a'b}{bb'}$. By definition of a,b and a',b', both are relatively prime to p. Hence bb' is relatively prime to p. Similarly, $ab'p^{c-c'} - a'b$ is relatively prime to p, provided that $c' \leq c$, which we have. Thus, $|x-y|_p = \left|p^{c'}\left(\frac{a}{b}p^{c-c'} - \frac{a'}{b'}\right)\right|_p = p^{-c'} \leq \max\left\{p^{-c'}, p^{-c}\right\} = \max\left\{|x-z|_p, |z-y|_p\right\}$.* ***2.*** *We need to define $|0|_p = 0$. Also, $p^{-c} \in \mathbb{R}^+$. Hence $d_p(x,y) \geq 0$ and* ***M1*** *is satisfied. For* ***M2****, Let $d_p(x,y) = 0$. But this means that $|x-y|_p = p^{-c} = 0$. This is where the definition $|0|_p = 0$ comes in and hence $x = y$. Conversely, if $x = y$, then x and y share the same factorization $\frac{a}{b}p^c$ and hence $|x-y|_p = 0$. For* ***M3****, let $x = \frac{a}{b}p^c$ and $y = \frac{a'}{b'}p^{c'}$ such that $c' \leq c$. Then, $x - y = p^{c'}\frac{ab'p^{c-c'} - a'b}{bb'}$ so that $|x-y|_p = p^{c'}$. Furthermore, $y - x = p^{c'}\frac{a'b - ab'p^{c-c'}}{bb'}$ so that $|y-x|_p = p^{c'}$ and hence $|x-y|_p = |y-x|_p$. Finally, for* ***M4****, let $d(x,z) = p^{-c'}$ and $d(z,y) = p^{-c}$. Since $\max\left\{p^{-c'}, p^{-c}\right\} \leq p^{-c'} + p^{-c}$, it follows that $|x-z+z-y|_p = |x-y|_p = d_p(x,y) \leq \max\left\{|x-z|_p, |z-y|_p\right\} \leq |x-z|_p + |z-y|_p = d_p(x,z) + d_p(z,y)$.* ***3.*** *Let $x,y,z \in \mathbb{Q}$. Assume that $d_p(x,y) \neq d_p(y,z)$. We may assume that $d_p(x,y) = |x-y|_p < |y-z|_p = d_p(y,z)$. Then,*

$$d_p(x,z) = |x-z|_p \leq \max\left\{|x-y|_p, |y-z|_p\right\} = |y-z|_p = d_p(y,z) \quad (4)$$

Also, from the assumption, $|x-y|_p < |y-z|_p \leq \max\left\{|x-z|_p, |x-y|_p\right\}$ by Strong Triangle Inequality and $\max\left\{|x-z|_p, |x-y|_p\right\} = |x-z|_p$. Thus, we *must have*

$$d_p(y,z) = |y-z|_p \leq |x-z|_p = d_p(x,z) \tag{5}$$

Then, (4) and (5) together imply that $d_p(y,z) = d_p(x,z)$. Thus for any three points, there will always be one side not equal to both of the other such that the other two will be equal. In other words, the points x, y, z define an isosceles triangle. ***4.*** *Let $r > 0$ be a real number and $x \in \mathbb{Q}$. Then, from the open set $N_r(x)$, consider $z \in C = (N_r(x))^c$. Consider the neighborhood $N_{r/2}(z)$ and*

let $y \in N_{r/2}(z)$. For any points x, y, z, we know that either $d_p(x,z) = d_p(x,y)$, $d_p(y,z) = d_p(x,y)$ or $d_p(x,z) = d_p(y,z)$ from Part (3). If $d_p(y,z) = d_p(x,y)$, then $d_p(x,y) < r/2$ so that $d_p(x,z) \leq d_p(x,y) + d_p(y,z) < r/2 + r/2 = r$ so that $z \notin C$, a contradiction. If $d_p(x,z) = d_p(y,z)$, then $d_p(x,z) < r/2$ so that $z \notin C$, again a contradiction. Thus, it can only be that $d_p(x,z) = d_p(x,y)$, then $d_p(x,y) \geq r$ so that $y \notin N_r(x)$. That is, $y \in C$ so that $N_{r/2}(z) \subset C$. Thus, there exists a neighborhood of z such that $z \in N_{r/2}(z) \subset C$. Since z was arbitrary, C is open.

Problem 25 *It is easy to prove that every open subset of $\mathbb{R}$ (with the standard topology) is the union of disjoint open intervals. The following problem shows that the analogue for closed sets is far from true. Let K be the set of real numbers x of the form*

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3}$$

where a_i is either 0 or 2. In other words numbers with no 1's in their base 3 expansion. This set is called the middle third Cantor set

1. *Show that $K \subset [0,1]$*
2. *Let s and k be positive integers. Show that $K \cap \left(\frac{3s+1}{3^k}, \frac{3s+2}{3^k}\right) = \emptyset$*
3. *Show that K is closed (Hint: show that K is the complement in $[0,1]$ of all the intervals from part 2)*
4. *Show that K contains no closed interval. In particular, this shows that K is not the union of closed intervals.*

Solution 26 ***1.*** *Clearly, for $a_i = 0$ for all i, then $0.a_1a_2... = 0$ and hence $0 \in K$. Next, the expression $0.222...$ in base 3 is 1. It may be concluded that $K = [0,1]$. However, since $a_i \neq 1$ for all i, many points in the closed interval are eliminated. Hence $K \subset [0,1]$.* ***2*** *For integers s, k such that $3s+1 > 3^k$, then $K \cap \left(\frac{3s+1}{3^k}, \frac{3s+2}{3^k}\right) = \emptyset$ since $K \subset [0,1]$. However, since $\frac{3s+1}{3^k} > 0$, we must have $\left(\frac{3s+1}{3^k}, \frac{3s+2}{3^k}\right) \subset [0,1]$ for integers s, k. Let $x = 0.a_1a_2...$ be a base 3 expression for some $x \in K \cap \left(\frac{3s+1}{3^k}, \frac{3s+2}{3^k}\right)$. That is, let $x = a_1\frac{1}{3} + a_2\frac{1}{3^2} + ...$ where $a_k = 0$ or 2 but not 1. However, $x \in \left(\frac{3s+1}{3^k}, \frac{3s+2}{3^k}\right)$ implies $a_k = 1$ with $3^{1-k}s = a_1\frac{1}{3} + a_2\frac{1}{3^2} + ...a_{k-1}\frac{1}{3^{k-1}}$, a contradiction to the fact that $x \in K$. Hence there is no such $x \in K \cap \left(\frac{3s+1}{3^k}, \frac{3s+2}{3^k}\right)$. Since x was arbitrary, it follows that*

*$K \cap \left(\frac{3s+1}{3^k}, \frac{3s+2}{3^k}\right) = \emptyset$. **3.** Let s, k be such that $\left(\frac{3s+1}{3^k}, \frac{3s+2}{3^k}\right) \subset [0,1]$. By (2), $K \cap \left(\frac{3s+1}{3^k}, \frac{3s+2}{3^k}\right) = \emptyset \implies [0,1] \setminus K = \bigcup_{s,k} \left(\frac{3s+1}{3^k}, \frac{3s+2}{3^k}\right)$. That is, the complement of K is open, since it is the union of arbitrary open sets. Hence K is closed. **4.** Let $[a,b] \subset [0,1]$, with $0 < a < b < 1$, be an arbitrary interval with base-3 expression $a = 0.a_1a_2...$ and $b = 0.b_1b_2....$ (that is, with $a_i, b_i \in \{0,1,2\}$). If any a_i (or b_i)$= 1$, then $a \notin K$ (or $b \notin K$). Since $a \neq b$, we can have let k be the smallest index such that $a_k \neq b_k$. Then, if $a_k = 0$ or 2 implies $b = 1$, a contradiction. If $b_k = 0$ or 2, then $a_k = 1$, another contradiction. Therefore, the interval $[a,b]$ is not in K.*

Theorem 27 *Let (X, d) be a metric space. Then, the following hold:*

1. *$\emptyset, X$ are open.*
2. *The union of an arbitrary number of open sets is open.*
3. *The intersection of finitely many open sets is open.*
4. *$N_\epsilon(x)$ is open*

Proof. 1. Vacuously, every point of $\emptyset$ is an interior point. Hence, it is open. Second, a ball of any radius, when and if constructed around any point of X will naturally be contained in X. Hence, X is open, too. For **2**, let A be an indexing set and suppose that U_α is open for each $\alpha \in A$ and $x \in \bigcup_{\alpha \in A} U_\alpha$. Then, there exists α such that $x \in U_\alpha$. Then, since U_α is open, there exists $\epsilon > 0$ such that $N_\epsilon(x) \subset U_\alpha \subset \bigcup_{\alpha \in A} U_\alpha$ and hence $\bigcup_{\alpha \in A} U_\alpha$ is open. Moving on to **3**, let U_i for $i \in \{1, ..., n\} = I_n$ be open. If $\bigcap_{i \in I} U_i = \emptyset$, then we are done. Assume otherwise. Then, there exists $x \in \bigcap_{i \in I} U_i$. That is, $x \in U_i$ for each i. By definition of open sets, there exists $N_{\epsilon_i}(x)$ such that $N_{\epsilon_i}(x) \subset U_i$. Set $\epsilon = \min_i \epsilon_i$. This exists since i is finite. Furthermore, since $\epsilon_i > 0$ for all i, then $\epsilon > 0$ and $\epsilon < \epsilon_i$ for all i. Hence neighborhood $N_\epsilon(x) \subset N_{\epsilon_i}(x) \subset U_i$ for each i. That is, $N_\epsilon(x) \subset \bigcap_{i \in I} U_i$. Since x was arbitrary, we are done. This cannot be extended indefinitely since $\min(r_1, r_2, ...)$ may be zero, giving us a singleton as the intersection, or even the empty set. As an example, consider the interval $\left(-\frac{1}{n}, \frac{1}{n}\right) = A_n$. Then, $\bigcap_n A_n = \{0\}$. **4**, let $x \in N_\epsilon(x)$. Then, clearly, $N_{\epsilon/2}(x) \subset N_\epsilon(x)$. ■

2 Introduction to Topological Spaces

The first three properties of **Theorem 27** are generalised to what's called a topology.

Definition 28 *Let X be a non-empty set and let $\tau \subset 2^X$. Then, (X, τ) is called a **topological space** if*

1. *$\emptyset, X \in \tau$*
2. *The union of an arbitrary number of elements of τ is in τ.*
3. *The intersection of finitely many members of τ is in τ.*

In this case, τ is said to be a **topology** on X.

Since members of τ are sets, we will take the liberty of calling such sets open. Thus, a set U will be called closed if its complement is open.

Example 29 *Let X be a countable set. Define $\tau = \{A : |A^c| < \infty\} \cup \{\emptyset\}$. Then, τ is a topology. The proof is easy: by assumption, $\emptyset \in \tau$. Since $|X^c| = 0$, $X \in \tau$. Let $U_\alpha \in \tau$ for α in some indexing set A. Then, since the arbitrary intersection of finite sets is finite, we must have $\bigcup_{\alpha \in A} U_\alpha = \left(\bigcap_{\alpha \in A} U_\alpha \right)^c \in \tau$. Finally, for a finite indexing set I, and $U_i \in \tau$ for $i \in I$, $\bigcap_{i \in I} U_i = \left(\bigcup_{i \in I} U_i \right)^c \in \tau$ because the finite union of finite sets is finite. This topology is called the **cofinite topology.***

Example 30 *In view of the **Theorem 27**, all metric spaces are topological spaces. A topological space which has an underlying metric will be called a **metric topology**.*

Example 31 *$\tau = \{\emptyset, X\}$ is the simplest example. This is called the **indiscrete topology**.*

Example 32 *On the other end of the spectrum of $\tau = \{\emptyset, X\}$, the collection $\left(X, 2^X\right)$ is also a topology, the proof of which is easy and will be skipped. This is called the **discrete topology**.*

The discrete topology is a metric topology and comes from the discrete metric. To see this, we first note that for any $x \in X$, $\{x\}$ is open because $d(x, x) = 0$ and the neighborhood $N_{1/2}(x) \subset \{x\}$. Next, let A be any subset of X and let $\epsilon \in (0, 1)$. Then, $\forall x \in A$, the ϵ-neighborhood $N_\epsilon(x) = \{x\} \subset A$ for all $\epsilon \in (0, 1)$. Hence A is open.

Problem 33 *List all topologies on the set $\{a, b\}$*

Solution 34 *First, $\{\emptyset, \{a, b\}\}$. Second, $\{\emptyset, \{a\}, \{a, b\}\}$. Third, $\{\emptyset, \{b\}, \{a, b\}\}$ and fourth $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.*

Problem 35 *Let X be an uncountable set and $\tau = \left\{U \in 2^X : U^c \text{ is countable}\right\} \cup \{\emptyset\}$. Then, τ is a topology. What can be said about τ if X is countable?*

Solution 36 *By definition, $\emptyset \in \tau$. Since $X \backslash X = \emptyset$ is countable, hence $X \in \tau$. Next, let $U_\alpha \in \tau$ for some $\alpha \in A$ where A is an arbitrary indexing set. Then, $X \backslash U_\alpha = U_\alpha^c$ is countable and so, $X \backslash \bigcup_{\alpha \in A} U_\alpha = X \cap \left(\bigcup_{\alpha \in A} U_\alpha \right)^c = X \cap \left(\bigcap_{\alpha \in A} U_\alpha^c \right) = \bigcap_{\alpha \in A} X \cap U_\alpha^c = \bigcap_{\alpha \in A} (X \backslash U_\alpha)$. Assuming the Axiom of Choice, since arbitrary intersection of countable sets is countable, we therefore have $\bigcap_{\alpha \in A} (X \backslash U_\alpha) = X \backslash \bigcup_{\alpha \in A} U_\alpha$ countable and thus $\bigcup_{\alpha \in A} U_\alpha \in \tau$. Finally, let I be a finite indexing set and let $U_i \in \tau$ for each i. If U_i is empty for some i, then $\bigcap_{i \in I} U_i = \emptyset$ and since $\emptyset \in \tau$ by definition, we have $\bigcap_{i \in I} U_i \in \tau$. Similar reasoning holds if $\bigcap_{i \in I} U_i = \emptyset$ but if $U_i \neq \emptyset$ for each $i \in I$. Assume that $\bigcap_{i \in I} U_i \neq \emptyset$. Then since finite union of countable sets is countable, it follows that $X \backslash \left(\bigcap_{i \in I} U_i \right) = X \cap \left(\bigcap_{i \in I} U_i \right)^c = X \cap \left(\bigcup_{i \in I} U_i^c \right) = \bigcup_{i \in I} (X \cap U_i^c)$ (by distribution of intersection over unions) is countable since each $X \cap U_i^c = X \backslash U_i$ is countable and thus $\bigcap_{i \in I} U_i \in \tau$.*

If X is countable, then every subset U of X is countable and, therefore, $X \backslash U$ is also countable. Hence the topology on X is the discrete topology.

Such a topology is called the **co-countable** topology.

As an aside, we prove that finite union of countable sets is countable

Proof. Let $A_1, A_2, ..., A_n$ be countable sets. Assume that they are disjoint for otherwise we can Construct $A_1', A_2', ..., A_n'$ such that $\bigcap_{i=1}^{n} A_i = \emptyset$ by letting $A_1' = A_1 \times \overbrace{\{1\} \times ... \times \{1\}}^{n-1 \text{ times}} = \{(a, 1, 1, ..., 1) : a \in A_1\}$, $A_2' = \{2\} \times A_2 \times \overbrace{\{2\} \times ... \times \{2\}}^{n-2 \text{ times}} = \{(2, a, 2, ..., 2) : a \in A_2\}$, ..., $A_n^{'} = \overbrace{\{n\} \times ... \times \{n\}}^{n-1 \text{ times}} \times A_n = \{(n, n, ..., n, a) : a \in A_n\}$. Now, since each A_i is countable, we can have $A_i = \left\{ a_1^{(i)}, a_2^{(i)}, ... \right\}$. Then, define $f : \bigcup_{i=1}^{n} A_i \longrightarrow \mathbb{N}$ as follows: $f(x) = nk + i$ if $x = a_k^{(i)}$. f is well-defined since if $x = y$, then $x, y \in A_i$. Then, $a_k^{(i)} = a_j^{(i)} \implies k = j \implies nk + i = nj + i \implies f\left(a_k^{(i)}\right) = f\left(a_j^{(i)}\right)$. Reversing these implications shows us that f is injective. Clearly, for every $m \in \mathbb{N}$ and any fixed n, by Euclidean algorithm, there exists unique k and i such that $m = nk + i$. Thus, f is surjective and that $\bigcup_{i=1}^{p} A_i$ is countable. ■

Problem 37 *Let (X, τ) be a topological space and let $Y = X \cup \{a\}$ where $a \notin X$. Let $\tau' = \{V \cup \{a\} : V \in \tau\} \cup \{\emptyset\}$. Show that (Y, τ') is a topological space*

Solution 38 ***T1*** *By definition, $\emptyset \in \tau'$. Also, since $X \in \tau$, we must have $X \cup \{a\} = Y \in \tau'$. For **T2**, let $U_\alpha \in \tau'$ for some index α in an indexing set A. Then, $\forall \alpha$, $\exists V_\alpha \in \tau$ such that $V_\alpha \cup \{a\} = U_\alpha$. Since τ is a topology, we must have $\bigcup_{\alpha \in A} V_\alpha \in \tau$. Hence $\left(\bigcup_{\alpha \in A} V_\alpha\right) \cup \{a\} = \bigcup_{\alpha \in A} (V_\alpha \cup \{a\}) = \bigcup_{\alpha \in A} U_\alpha \in \tau'$, by definition of τ'. **T3**, Similarly, if $U_i \in \tau'$ for index i in finite indexing set I, then, $\forall i$, $\exists V_i \in \tau$ such that $V_i \cup \{a\} = U_i$. Since τ is a topology, we must have $\bigcap_{i \in I} V_i \in \tau$. Hence $\left(\bigcap_{i \in I} V_i\right) \cup \{a\} = \bigcap_{i \in I} (V_i \cup \{a\}) = \bigcap_{i \in I} U_i \in \tau'$. Hence τ' is a topology on Y.*

We are now lead to our first theorem.

Theorem 39 *Let (X, τ) be a metric topology with $|X| > 1$. Then, $\forall x \in X$, $\{x\}^c$ is open. That is, every singleton is closed.*

Proof. Let d be the underlying metric on τ. For any $y \in \{x\}^c$, define $\alpha = d(x, y)$. Since $x \neq y$, we must have $\alpha > 0$. Let $z \in N_{\alpha/2}(y) = \{z : d(z, y) < \alpha/2\}$. Then, $d(z, y) < \alpha/2 < \alpha = d(x, y)$. If $z = x$, then $d(z, x) = 0$ and $\alpha < 0$, a contradiction. Hence $z \neq x$ and $z \in \{x\}^c$. ■

Corollary 40 *Let $|X| > 1$. Then, the indiscrete topology (X, τ_I) is not a metric topology.*

Proof. Let $y \in X$. Then, $\{y\}^c \subset X$, a proper subset of X and thus not open, since the only open subsets of X are improper. Hence $\{y\}^c$ is not open. In other words, no singleton is closed, since y was arbitrary. Therefore, (X, τ_I) is not a metric topology. ■

There is a high-brow sounding name for a topology on X on which every singleton is closed and its called a T_1**-space**. Thus, we could say that all metric topologies are T_1 and that the indiscrete topology is not T_1. If $|X| = 1$, then for $y \in X$, $\{y\}^c = \emptyset$ is open and hence the above theorems fail to hold. The only metric we can have on a singleton is the silly metric $d(x, y) = 0$ for all $x, y \in \{x\} = X$, which we won't worry about much.

Problem 41 *Let (X, τ) be a topological space. Show that (X, τ) is T_1 if and only if every finite set is closed.*

Solution 42 *By definition, (X, τ) is T_1 if and only if every singleton $\{x\}$ is closed for $x \in X$. If $|X| = 1$, then the result is trivial. Assume that $|X| > 1$ and let X be T_1 and A be a finite set. If $|A| < 1$, then $A = \emptyset$ and there's nothing left to prove. If $|A| = 1$, then $X \backslash A$ is open, by definition of T_1 and therefore A is closed. Assume $|A| = n$ for $n \geq 1$. Then, we can label points as $\{x_1, ..., x_n\}$.*

From $A = \bigcup_{i=1}^{n} \{x_i\}$, *we can have* $A^c = \bigcap_{i=1}^{n} \{x_i\}^c$ *by De Morgan's Law. Since* $\{x_i\}$ *is closed for each* i, *we must have* $\{x_i\}^c$ *open. Since finite intersection of open sets in a topology is open, we must have* A^c *open. In other words,* A *must be closed. Conversely, let* A *be a finite and closed set. Again, if* $|A| < 1$, *then* $A = \emptyset$ *and there's nothing left to prove. If* $|A| = 1$, *then every singleton is closed and* X *is* T_1. *Assume* $|A| > 1$. *Let* $x, y \in A$ *with* $x \neq y$ *and let* $U = \{y\}^c$. *Then,* $x \in U$ *so that* U *is non-empty. Since every finite set is closed,* $\{y\}$ *must be closed. It follows that* X *is* T_1.

In order to prove that the cofinite topology is not a metric topology, we will need a little more machinery.

Definition 43 *Let* (X, τ) *be a topological space. Then,* (X, τ) *is said to be a* ***Hausdorff space*** *(equivalently,* τ *is Hausdorff) if* $\forall x, y \in X$ *and* $x \neq y$, *there exists* $U, V \in \tau$ *such that* $x \in U$, $y \in V$ *and* $U \cap V = \emptyset$.

Problem 44 *Every Hausdorff space is a* T_1 *space*

Solution 45 *Let* x, y *be distinct points in a Hausdorff space* (X, τ). *Consider* $x \in X \setminus \{y\}$. *Then, there exists* U, V *such that* $x \in U$, $y \in V$ *such that* $U \cap V = \emptyset$. *Clearly,* $x \in U \subseteq X \setminus \{y\}$. *This shows that* $X \setminus \{y\}$ *is open, or that* $\{y\}$ *is closed. Since* y *(and* x*) was (were) arbitrary, therefore* X *is* T_1.

Theorem 46 *Let* (X, τ) *be a metric topology with* $|X| > 1$ *and let* d *be a metric on* X. *Then,* τ *is Hausdorff*

Proof. Let x, y be distinct points in X. Then, $d(x, y) > 0$. Let $\alpha = d(x, y)$. Then, the sets $U = N_{\alpha/2}(x)$ and $N_{\alpha/2}(y)$ are disjoint for otherwise, let $z \in U \cap V$, then $d(z, x) < \alpha/2$ and $d(z, y) < \alpha/2$. Then, $d(x, y) \leq d(x, z) + d(y, z) \implies \alpha < \alpha/2 + \alpha/2$, a contradiction. Needless to say, $x \in U$ and $y \in V$. ■

Theorem 47 *Let* X *be a non-empty, finite set. Then, the cofinite topology* (X, τ) *is the same as the discrete topology.*

Proof. Since X is finite, every subset of X is finite and so, the complement of every subset of X is finite. Thus, $2^X = \left\{A \in 2^X : A^c \text{ is finite}\right\} = \tau$. ■

Theorem 48 *Let* (X, τ) *be the cofinite topology. If* $|X| = \infty$, *then the cofinite topology is not a Hausdorff Space.*

Proof. Let $x, y \in X$ be two distinct points and assume that there exist two open sets U, V such that $U \cap V = \emptyset$. Then, $U^c \cup V^c = X$. However, since U^c and V^c are finite, their union cannot possibly equal to an infinite set. Thus, no two open sets are disjoint, implying that (X, τ) is not Hausdorff. ■

Corollary 49 *Cofinite topology is not a metric topology for* $|X| = \infty$.

Proof. Since Cofinite topology is not Hausdorff, and by **Theorem 46**, the cofinite topology is not a metric topology. ■

2.1 Construction of Topologies

Let (X, τ_1) and (X, τ_2) be two topologies. Then, clearly, $\tau_1 \cap \tau_2$ is also a topology on X. This is how new topologies on X can be formed. However, the union of two topologies, in general, is not a topology. For example, if $X = \{a, b, c\}$ and $\tau_1 = \{X, \emptyset, \{a\}\}$ and $\tau_2 = \{X, \emptyset, \{b\}\}$, then $\tau_1 \cup \tau_2 = \{X, \emptyset, \{a\}, \{b\}\}$ does not form a topology on X as $\{a, b\} = \{a\} \cup \{b\}$ is not open in $\tau_1 \cup \tau_2$.

From set-theoretic properties, it is known that $\tau_1 \cap \tau_2 \subset \tau_1$ and that $\tau_1 \cap \tau_2 \subset \tau_2$. The intersection of two topologies thus results in a "smaller" topology. Since a topology on a set X is just a collection, and we can compare collections by using the relation $\subset$, it makes sense to talk about comparison of topologies.

Let X be a non-empty set and let τ_1 and τ_2 be two topologies on X. Then, τ_1 is said to be **finer** or **stronger** than τ_2 if $\tau_2 \subset \tau_1$. Equivalently, τ_2 is said to be **weaker** or **coarser** than τ_1, which is psychologically more appealing! If a topology is weaker, then it has less open sets, by definition of set-containment. Any given topology on X is always stronger than the indiscrete topology but weaker than the discrete topology. In other words, the indiscrete topology is always coarser than any other topology (on the same universal set). On the other end of the spectrum, every subset of X is open in the discrete topology so that the discrete topology is the finest topology a set can have.

If we are interested in certain open sets, can we get to the smallest topology in which our sets of interest are open? To be precise, let $A \subset 2^X$ contain sets of our interest. What is the smallest topology on X which contains A? To the point of annoying precision, what is the smallest set τ such that $A \subset \tau$ and τ is a topology on X? One very predictable way is to use the intersection of all topologies which contain A. This is called the topology generated by A and is denoted by $\tau(A)$.

Theorem 50 *$\tau(A)$ is the smallest topology on X containing A*

Proof. Let I be an arbitrary indexing set and let $\tau_i \subset 2^X$ be a topology on X such that for each $i \in I$ such that $A \subset \tau_i$ for each i. Let $\tau(A) = \bigcap_{i \in I} \tau_i$. $\tau(A)$ is non-empty since, by definition, $A \subset \tau_i$ for each i and so that $A \subset \tau(A)$. Since $\emptyset, X \in \tau_i$ for each i, then $\emptyset, X \in \tau(A)$. Let U_γ be sets in $\tau(A)$ for index $\gamma \in \Gamma$ where Γ is an indexing set. Then, $U_\gamma \in \tau_i$ for each $i \implies \bigcup_{\gamma \in \Gamma_i} U_\gamma \in \tau_i$ for some indexing set Γ_i for each $i \implies \bigcap_{i \in I} \bigcup_{\gamma \in \Gamma_i} U_\gamma \in \bigcap_{i \in I} \tau_i \implies \bigcup_{\gamma \in \Gamma} U_\gamma \in \tau(A)$. Similarly, the finite intersection of elements of $\tau(A)$ is in $\tau(A)$. To see that it is the smallest one containing A, let $\mathcal{A} = \{\varrho_A \subset 2^X : A \subset \varrho_A \text{ and } \varrho_A \text{ is a topology on } X\}$ and let $\varrho \in \mathcal{A}$ such that $\varrho \subset \tau(A)$. By definition, $\tau(A) = \bigcap_{\varrho_A \in \mathcal{A}} \varrho_A \subset \varrho$. Thus, $\varrho = \tau(A)$. ■

However, a more practical construction is done by using what's called a subbasis. Before giving its definition, we motivate its construction: let B be the collection of all finite intersection of elements of A. This is non-empty since A is non-empty. Let A^* be the collection of all unions of elements in B, including $\emptyset$.

Theorem 51 $\forall A \subset 2^X$, $A^* = \tau(A)$

Proof. We need to show that A^* is a topology. From definitions of A and B, we have $A \subset B$ since single-intersections can produce elements of A itself. Furthermore, single-unions of B can produce B itself so that we have the inclusions $A \subset B \subset A^*$. Thus, A^* is non-empty since A is non-empty. By definition, $\emptyset \in A^*$. By convention, the empty intersection is defined to be X so that $X \in A^*$.

Next, let $U_i \in A^*$ where $i \in I$ for an arbitrary indexing set I. Then, for each i, U_i is a union of elements in B. Let $U_i = \bigcup\limits_{\beta} U_{i,\beta}$ where β is some index. Now, $\bigcup\limits_{i} U_i = \bigcup\limits_{i} \left(\bigcup\limits_{\beta} U_{i,\beta} \right)$. Since elements of A^* are all unions of elements of B and elements of B are finite intersections of elements of A, it follows that $\bigcup\limits_{i} U_i$ is a union of finite intersections of elements of A. Hence **T2** holds. Let $U, V \in A^*$. Then, $U = \bigcup\limits_{\alpha \in A} U_\alpha$ and $V = \bigcup\limits_{\beta \in B} V_\beta$ where U_α and V_β are results of finite intersections of elements of A for each α and β. Then, $U \cap V = \bigcup\limits_{\alpha \in A, \beta \in B} U_\alpha \cap V_\beta$. $U_\alpha \cap V_\beta$ is finite since each is a finite intersection $\implies U \cap V \in A^*$. We can then use induction to show that the same holds for finite intersections. Hence **T3** is satisfied, so that A^* forms a topology.

Thus, $\tau(A) \subset A^*$. To show the converse, we know that $A \subset \tau(A) \implies B \subset \tau(A)$ since B is simply formed by taking finite intersections of elements of A and $\tau(A)$ is a topology. Finally, $A^* \subset \tau(A)$ since, again, A^* is a collection of unions of elements of B including $\emptyset$ and $\tau(A)$ is a topology. ■

Definition 52 *Let (X, τ) be a topological space. If (X, τ) is a topological space and $A \subset 2^X$ such that $A^* = \tau$, then A is called* ***subbasis*** *for τ. A collection (sub-basis?) $B \subset 2^X$ is a* ***basis*** *for τ if*

B1. *$\forall x \in X$, $\exists U \in B$ such that $x \in U$*

B2. *If $U, V \in B$ and $x \in U \cap V$, then $\exists W \in B$ such that $x \in W \subset U \cap V$*

Every basis is a subbasis since finite intersections of elements of B are in B and unions of finite intersections of elements of B generate the same topology as the topology generated by B:

Lemma 53 *Let B be a basis of topology τ. If U is an open set, then $U = \bigcup\limits_{\alpha \in A} U_\alpha$, where $U_\alpha \in B$ for some indexing set A.*

Proof. Let U be an open subset of X and $x \in U \subset X$. By **B1**, $\exists U_x$ such that $x \in U_x$. Clearly, $U_x \subset U$ so that $\bigcup_{x\in U} U = U \supset \bigcup_{x\in U} U_x$. Conversely, let $x \in U$. Then, clearly, $x \in \bigcup_{x\in U} U_x$ ■

Since unions of finite intersections of elements of B form τ and for a fixed x, U_x alone may be considered as as a finite intersection of itself with itself, then $\bigcup_{x\in U_x} U_x = U_x$ is an open set.

Example 54 *Let (X, d) be a metric space with topology τ. Then,*

$$B = \{N_\epsilon(x) : x \in X \text{ and } \epsilon > 0\}$$

is a basis since $\forall x \in X$, we have $x \in N_\epsilon(x)$. Also, if $x \in N_{\epsilon_1}(a) \cap N_{\epsilon_2}(b)$, then $d(x,a) < \epsilon_1$ and $d(x,b) < \epsilon_2$. Now let $\epsilon = \min\{\epsilon_1 - d(x,a), \epsilon_2 - d(x,b)\}$. Then, for $y \in N_\epsilon(x)$, $d(y,a) \leq d(y,x) + d(x,a) < \epsilon + d(x,a) < \epsilon_1 - d(x,a) + d(x,a) = \epsilon_1$ so that $y \in N_{\epsilon_1}(a)$. Similarly, $d(y,b) \leq d(y,x) + d(x,b) < \epsilon + d(x,b) < \epsilon_2 - d(x,b) + d(x,b) = \epsilon_2$ so that $y \in N_{\epsilon_1}(a) \cap N_{\epsilon_2}(b)$. That is, $N_\epsilon(x) \subset N_{\epsilon_1}(a) \cap N_{\epsilon_2}(b)$ and hence $\exists N_\epsilon(x)$ such that $x \in N_\epsilon(x) \subset N_{\epsilon_1}(a) \cap N_{\epsilon_2}(b)$

Example 55 *Let $X = \mathbb{R}$ with standard topology. Then,*

$$B = \left\{N_\epsilon(x) : x \in \mathbb{Q} \text{ and } \epsilon \in \mathbb{Q}^+\right\}$$

is a basis. Let $r \in \mathbb{R}$. If r is positive, then by Archimedes axiom, there exists n such that $r < 1n = n$. Clearly, for positive r, $-n < r$ so that $|r| < n$. If r is negative, then there exists an m such that $-r < 1m \implies -m < r$, and, of course, $r < m$ since r is negative so that $|r| < m$. In either case, $|r|$ is less than an integer (rational number). If $r = 0$, then $|r| < 1$ and in this case, interval $N_\epsilon(0)$ is our required element of basis for any $\epsilon \in \mathbb{Q}^+$. Hence for all $r \in \mathbb{R}$, there exists rationals a, b such that $a < r < b$. From this, we have that $0 < \frac{r-a}{2} < \frac{b-a}{2}$, $\frac{r-b}{2} < 0$. Let $x = \frac{a+b}{2}$ and $\epsilon = b - a$. Then, $r \in N_\epsilon(x)$ because

$$\begin{aligned} d(r,x) &= |r-x| = \left|r - \frac{a+b}{2}\right| = \left|\frac{r-a}{2} + \frac{r-b}{2}\right| \\ &\leq \left|\frac{r-a}{2}\right| + \left|\frac{r-b}{2}\right| = \frac{r-a}{2} + \frac{b-r}{2} \\ &< \frac{b-a}{2} + \frac{b-r}{2} < \frac{b-a}{2} + \frac{b-a}{2} = b - a = \epsilon \end{aligned}$$

*This construction satisfies **B1**. Now let $N_{\epsilon_1}(x_1)$ and $N_{\epsilon_2}(x_2)$ be two elements of B and $x \in N_{\epsilon_1}(x_1) \cap N_{\epsilon_2}(x_2)$. For $\epsilon = \min\{\epsilon_1 - d(x,x_1), \epsilon_2 - d(x,x_2)\}$ so that for $y \in N_\epsilon(x)$, $d(y,x_1) \leq d(y,x) + d(x,x_1) < \epsilon + d(x,x_1) < \epsilon_1 - d(x,x_1) + d(x,x_1) = \epsilon_1$ so that $y \in N_{\epsilon_1}(x_1)$. Similarly, $d(y,x_2) \leq d(y,x) + d(x,x_2) < \epsilon + d(x,x_2) < \epsilon_2 - d(x,x_2) + d(x,x_2) = \epsilon_2$ so that $y \in N_{\epsilon_1}(x_1) \cap N_{\epsilon_2}(x_2)$. That is, $N_\epsilon(x) \subset N_{\epsilon_1}(x_1) \cap N_{\epsilon_2}(x_2)$ and hence $\exists N_\epsilon(x)$ such that $x \in N_\epsilon(x) \subset N_{\epsilon_1}(x_1) \cap N_{\epsilon_2}(x_2)$*

Example 56 *In discrete topology, $B = \{\{x\} : x \in X\} \cup \{\emptyset\}$. By definition, $\forall x \in X$, $\{x\} \in B$ so that $\exists U \in B$ such that $x \in U$. If $U = \{x\} = V = \{y\}$, then the condition is trivially satisfied. For $x \neq y$, then $W = \emptyset$ so that **B2** is satisfied.*

Lemma 57 *Let B be a basis for topology τ. If $U = \bigcap_{i=1}^{n} U_i$ where $U_i \in B$, then $U = \bigcup_{\alpha \in A} U_\alpha$, where $U_\alpha \in B$ for some indexing set A.*

Proof. By **B2**, $U_i \in B \implies U \in B$. Thus, U is an open set and the **Lemma 53** may be applied. ■

Theorem 58 (Criteria for basis-ness) *Let (X, τ) be a topological space and $\mathcal{C} \subset \tau$ be a collection of open sets such that $\forall$ open sets U and $\forall x \in U$, $\exists V \in \mathcal{C}$ such that $x \in V \subset U$. Then, $\mathcal{C}$ is a basis for τ.*

Proof. For **B1**, let $x \in X$. Since X is open, $\exists U \in \mathcal{C}$ such that $x \in U \subset X$. For **B2**, let U_1, $U_2 \in \mathcal{C}$ and $x \in U_1 \cap U_2$. Then, $U_1 \cap U_2$ is open $\implies \exists U_3$ such that $x \in U_3 \subset U_1 \cap U_2 \implies \mathcal{C}$ is a basis. To show that $\mathcal{C}$ is a basis for τ, let $\mathcal{C}^*$ be the topology generated by $\mathcal{C}$. If U is an open set, $x \in U$, then $\exists U_x \in \mathcal{C}$ such that $x \in U_x \subset U$ and $U = \bigcup_{x \in U} U_x$ by **Lemma 53**, so $U \in \mathcal{C}^*$ so that $\tau \subset \mathcal{C}^*$. If $U \in \mathcal{C}^*$, then $U = \bigcup_{\alpha \in A} U_\alpha$ where $U_\alpha \in \mathcal{C}$ by definition of topology generated by $\mathcal{C}$. Since $\mathcal{C} \subset \tau$, we must have $U \in \tau$, we must have $\mathcal{C}^* \subset \tau$. ■

Thus, B is a basis for τ if for every open set U, $U = \bigcup_{\alpha \in A} U_\alpha$ where $U_\alpha \in B$.

Following allows us to compare topologies by comparing their bases:

Theorem 59 *Let X be a non-empty set and let (X, τ) and (X, τ') be two topologies on X, B be basis for topology τ and B' be basis for topology τ'. Then, the following are equivalent*

1. *τ' is finer than τ*
2. *$\forall x \in X$, $\forall U \in B$ with $x \in U$, $\exists U' \in B'$ such that $x \in U' \subset U$*

Proof. $(1 \implies 2)$

Suppose τ' is finer than τ. Let $x \in X$ and $x \in U \in B$ be an open set in τ. Then, U is open in τ' and so $U = \bigcup_{\alpha \in A} U_\alpha$ where $U_\alpha \in B'$. Since $x \in U$, there exists $\alpha \in A$ such that $x \in U_\alpha \implies U_\alpha \subset U$.

$(2 \implies 1)$

Let U be an open set in τ. To show that U is an open set in τ', write $U = \bigcup_{\alpha \in A} U_\alpha$ where $U_\alpha \in B$. $\forall x \in U$, there exists α such that $x \in U_\alpha \subset U \implies$

$\exists U'_\alpha \in B'$ such that $x \in U'_\alpha \subset U_\alpha$. Then, $U \supset \bigcup_{\alpha \in A} U'_\alpha \supset \bigcup_{\alpha \in A} U_\alpha = U$. Thus, $U = \bigcup_{\alpha \in A} U'_\alpha \implies U$ is open in τ'. ■

Example 60 *l^1 metric and l^2 metric are equivalent, which we will prove using **Theorem 59**. l^2 is finer than l^1 since for any basis element of l^1, say $N_\epsilon^{l^1}(0)$, we always have $N_{\frac{\epsilon}{\sqrt{2}}}^{l^2}(0) \subset N_\epsilon^{l^1}(0)$. Conversely, l^1 is finer than l^2 since $N_\epsilon^{l^1}(0) \subset N_\epsilon^{l^2}(0)$ by default.*

More generally, we have the following:

Problem 61 *Let $p > 0$ and define $d_p : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by*

$$d_p\left((x_1, ..., x_n), (y_1, ..., y_n)\right) = \left(|x_1 - y_1|^p + ... + |x_n - y_n|^p\right)^{1/p}$$

Show that if $p, p' \geq 1$, then the l^p and $l^{p'}$ metrics induce the same topology.

Solution 62 *Let $\mathbf{x} \in \mathbb{R}^n$. Then, for $\epsilon > 0$, define $N_{\epsilon,p}(\mathbf{x}) = \{\mathbf{y} : d_p(\mathbf{x}, \mathbf{y}) < \epsilon\}$ for some $p \geq 1$. To show that the collection $\mathcal{C} = \{N_{\epsilon,p}(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\}$ form a basis for the topological space generated by $d_p(\mathbf{x}, \mathbf{y})$, we will use the fact that for each open set U of $\mathbb{R}^n$ and each $\mathbf{x} \in U$, there is an element $N_{\epsilon,p}(\mathbf{x})$ of $\mathcal{C}$ such that $\mathbf{x} \in N_{\epsilon,p}(\mathbf{x}) \subset U$. This can be satisfied by choosing $\epsilon = \frac{1}{2}\sup\{d(\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in U\}$. If $1 \leq p' \leq p < \infty$, then $|x_i - y_i|^{p'} \leq |x_i - y_i|^p$ so that $\sum_{i=1}^n |x_i - y_i|^{p'} \leq \sum_{i=1}^n |x_i - y_i|^p$ and hence $d_p(\mathbf{x}, \mathbf{y}) \leq d_{p'}(\mathbf{x}, \mathbf{y})$. By this comparison, we note that $N_{\epsilon,p'}(\mathbf{x}) \subseteq N_{\epsilon,p}(x)$. Clearly, $N_{\frac{\epsilon}{\sqrt{2}},p}(\mathbf{x}) \subseteq N_{\epsilon,p'}(\mathbf{x})$. It follows that the topology generated by $d_{p'}(\mathbf{x}, \mathbf{y})$ is finer than that for $d_p(\mathbf{x}, \mathbf{y})$ and conversely. Hence both topologies are the same. For the case of $p = \infty$, it suffices to compare the basis for the topology generated by $d_\infty(\mathbf{x}, \mathbf{y})$ with $d_1(\mathbf{x}, \mathbf{y})$. Clearly, $\max\{|x_i - y_i| : 1 \leq i \leq n\} \leq \sum_{i=1}^n |x_i - y_i|$ and hence $d_\infty(\mathbf{x}, \mathbf{y}) \leq d_1(\mathbf{x}, \mathbf{y})$ so that $N_{\epsilon,\infty}(\mathbf{x}) \subseteq N_{\epsilon,1}(\mathbf{x})$. Hence the topology generated by $d_1(\mathbf{x}, \mathbf{y})$ is coarser than the topology generated by $d_\infty(\mathbf{x}, \mathbf{y})$. To see the converse, we note that $N_{\epsilon,1}(\mathbf{x}) \subseteq N_{n\epsilon,\infty}(\mathbf{x})$ and we are done.*

Example 63 *Consider the topologies on $\mathbb{R}$ with basis $\mathcal{B} = \{(a, b) : a < b\}$ for the standard topology, $\mathcal{B}' = \{[a, b) : a < b\}$, the lower limit topology and $\mathcal{B}'' = \{(a, b) - K : a < b\} \cup B$ for the K-topology where $K = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Then, for each $x \in (a, b) \in \mathcal{B}$, we have $x \in [x, b) \subset (a, b)$ so that the standard topology is coarser the lower limit topology. Also, for each $x \in (a, b)$, we have $x \in (a, b) - K \subset (a, b)$ if $x \neq \frac{1}{n}$ for any n and $x \in (a, b) \subset (a, b)$ otherwise. Thus, the K-topology is also finer than the standard topology. However, K-topology and lower limit topology are not comparable.*

From hereon, we will still be studying how topological spaces arise and deform but for that, we need a little more machinery.

2.2 Closures and Interiors

Let (X, τ) be a topological space and let A be a subset of X. Then, $Int\,(A)$ is the union of all open sets contained in A and $\overline{A}$ is the intersection of all closed sets containing A. Thus, by definition, $Int\,(A) \subset A \subset \overline{A}$. To be more precise, let $x \in Int\,(A)$. Then, since $Int\,(A) = \bigcup_{\alpha \in \Gamma} U_\alpha$ with $U_\alpha \subset A$ for each α in an indexing set Γ, we must have $x \in \bigcup_{\alpha \in \Gamma} U_\alpha \implies x \in U_\alpha$ for some α in general and in A in particular (since $\forall \alpha, U_\alpha \subset A$). Hence $x \in A$ so that $Int\,(A) \subset A$. Furthermore, for $x \in A$, $\overline{A} = \bigcap_{\alpha \in \Gamma} U_\alpha$ where U_α is a closed set containing A for each α, then $x \in A \subset U_\alpha$ for each $\alpha \implies x \in \overline{A}$.

Proposition 64 *Let (X, τ) be a topological space and let A be a subset of X. Then, A is open if and only if $A = Int\,(A)$.*

Proof. ($\Longrightarrow$) Let $\mathcal{A}$ be the collection of open sets contained in A. Then, since $A \subset A$, we have that $A \in \mathcal{A}$ and $Int\,(A) = \bigcup_{U \in \mathcal{A}} U = A$.

($\Longleftarrow$) $Int\,(A)$ is open since it is a union of open sets. Since $Int\,(A) = A$, therefore A is open. ■

Proposition 65 *Let (X, τ) be a topological space and let A be a subset of X. Then, A is closed if and only if $A = \overline{A}$.*

Proof. ($\Longrightarrow$) Let $\mathcal{A}$ be the collection of closed sets containing A. Then, since $A \subset A$, we have that $A \in \mathcal{A}$ and $\overline{A} = \bigcap_{U \in \mathcal{A}} U = A$.

($\Longleftarrow$) First, we notice that $\overline{A} = \bigcap_{U \in \mathcal{A}} U = A \iff \overline{A}^c = \bigcup_{U \in \mathcal{A}} U^c = A^c$ with U^c open. Next, $\bigcup_{U \in \mathcal{A}} U^c$ is open since it is the arbitrary union of open sets. Thus, A^c is open or that A is closed. ■

In particular, this means that $\overline{A}$ is closed.

Proposition 66 *Let (X, τ) be a topological space and let A be a subset of X. Then, $x \in \overline{A} \iff$ every neighborhood of x intersects A*

Proof. We prove the contrapositive: that, $x \notin \overline{A} \iff \exists$ neighborhood U of x such that $A \cap U = \emptyset$.

($\Longrightarrow$) Let $x \notin \overline{A}$. Then, $x \in U = X - \overline{A}$ where U is some open set. Thus, U is a neighborhood of x. Since $A \subset \overline{A}$, $\overline{A} \cap U = \emptyset$ and $A \cap U \subset \overline{A} \cap U$, it follows that $A \cap U = \emptyset$.

($\Longleftarrow$) Let $x \in X$ and assume that $\exists$ neighborhood U of x such that $A \cap U = \emptyset \implies A \subset X - U$. Since U is open, we must have $X - U$ closed so that $\overline{A} \subset X - U$, by definition of $\overline{A}$ so that if $x \in U$, then $x \notin \overline{A}$. ■

This can even be reduced to the basis.

Proposition 67 *Let (X, τ) be a topological space and let A be a subset of X and B be a basis for τ. Then, $x \in \overline{A} \iff$ every basis element containing x intersects A*

Proof. Again, the contrapositive "$x \notin \overline{A} \iff \exists$ neighborhood U of x such that $A \cap U = \emptyset$" is proved:

($\Longrightarrow$) Let $x \notin \overline{A}$. Then, $x \in U = X - \overline{A}$ where U is some open set. Thus, U is a neighborhood of x. Since U is open $U = \bigcup\limits_{\alpha \in \mathcal{A}} U_\alpha$ where $U_\alpha \in B$ for each α in an indexing set $\mathcal{A}$. Let $\mathcal{A}'$ be an indexing set such that $x \in U_\alpha$ for $\alpha \in \mathcal{A}'$. Since $A \subset \overline{A}$, it follows that $A \cap U_\alpha = \emptyset$ for for $\alpha \in \mathcal{A}'$.

($\Longleftarrow$) Let $x \in X$ and assume that $\exists U \in B$ with $x \in U$ such that $A \cap U = \emptyset \implies A \subset X - U$. Since U is a basis element, open, we must have $X - U$ closed so that $\overline{A} \subset X - U$ so that if $x \in U$, then $x \notin \overline{A}$. ■

Proposition 68 $\emptyset = \overline{\emptyset}$*,* $\overline{A \cup B} = \overline{A} \cup \overline{B}$ *and* $\overline{\overline{A}} = \overline{A}$

Proof. Since X is open in any topology, then $X^c = \emptyset$ is closed. By **Proposition 65**, $\emptyset = \overline{\emptyset}$

$A \subset \overline{A}$ and $B \subset \overline{B}$ implies $A \cup B \subset \overline{A} \cup \overline{B}$. Since $\overline{A}$ and $\overline{B}$ are both closed, $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B = C$, say. Thus, $\overline{C} \subset \overline{A} \cup \overline{B}$ by definition or that $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. To show the converse, let $x \in \overline{A} \cup \overline{B}$. Then, $x \in \overline{A}$ or $x \in \overline{B}$. If $x \in \overline{A}$, then every neighborhood U of x intersects A and so intersects $A \cup B$. Thus, $x \in \overline{A \cup B}$. Similarly, if $x \in \overline{B}$, then every neighborhood U of x intersects B and so intersects $B \cup A$. **Proposition 67**, $x \in \overline{A \cup B}$. In either case, $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$ so that $\overline{A} \cup \overline{B} = \overline{A \cup B}$. Finally, combining **Proposition** 65 and the fact that $\overline{A}$ is closed gives us $\overline{\overline{A}} = \overline{A}$. ■

Theorem 69 *$Int\,(A)$ is the largest open set contained in A*

Proof. Clearly, $Int\,(A) \subseteq A$ by definition. To prove that this is the largest such set, assume that there exists another open set O such that $Int\,(A) \subseteq O \subseteq A$. Then, let $x \in M = O - Int\,(A) \implies x$ is not an interior point of A. In particular, it is not an interior point of O. Since x is arbitrary, therefore O is not an open set, a contradiction. Hence $x \notin M = O - Int\,(A)$ so that $M = \emptyset$, establishing the required theorem. ■

Proposition 70 *$Int\,(X) = X$, $Int\,(\emptyset) = \emptyset$, $Int\,(A \cap B) = Int\,(A) \cap Int\,(B)$ and $Int\,(Int\,(A)) = Int\,(A)$*

Proof. Clearly, $Int\,(X) \subset X$ for any subset of the universal set X, in general, and for $Int\,(X)$ in particular. For reverse inclusion, let $x \notin Int\,(X)$. Then, x is not a member of any open set containing $Int\,(X)$. In particular, since X is also an open set containing $Int\,(X)$, it follows that $x \notin X$. That is, $x \notin Int\,(X) \implies x \notin X$ the contrapositive of which is $x \in X \implies x \in Int\,(X)$ and hence $X \subset Int\,(X)$. For the second part, the empty set is a subset of every set. Thus, $\emptyset \subset I\,(\emptyset)$ holds for the particular set $I\,(\emptyset)$. To show the converse, let

$x \in I(\emptyset)$. Then, $x \in \bigcup_{\alpha \in A} U_\alpha$ with $U_\alpha \subset \emptyset$ for each α and U_α open. This is only possible if $U_\alpha = \emptyset$ for each α so that $\emptyset = \bigcup_{\alpha \in A} U_\alpha$ and hence $x \in \emptyset = \bigcup_{\alpha \in A} U_\alpha$, concluding that $I(\emptyset) \subset \emptyset$. Third, let $A, B \in 2^X$ and let $x \in I(A \cap B) \iff x \in \bigcup_{\alpha \in A} U_\alpha$ with U_α open and $U_\alpha \subset A \cap B$ for each α. Since $A \cap B \subset A$ and $A \cap B \subset B$, it follows that $x \in \bigcup_{\alpha \in A} U_\alpha$ with U_α open and $U_\alpha \subset A \cap B \iff U_\alpha \subset A$ and $U_\alpha \subset B$. Thus, U_α is an open set contained in A and B for all $\alpha \iff$ by definiton, means that $x \in I(A)$ and $I(B) \iff x \in I(A) \cap I(B)$. Finally, for any set A, $I(A) \subset A$. Hence $I(I(A)) \subset I(A)$. To show the reverse inclusion, let $x \in I(A)$. Since $I(A)$ is an open set and, by definition of interior, $I(I(A)) = \bigcup_{\gamma \in \Gamma} U_\gamma$ with $U_\gamma \subset I(A)$ for each γ where U_γ is an open set and Γ is an indexing set, it follows that there exists an open set $U_{\gamma'}$ for some $\gamma' \in \Gamma$, a neighborhood of x, such that $x \in U_{\gamma'} \subset I(A)$. Thus, $x \in \bigcup_{\gamma \in \Gamma} U_\gamma = I(I(A))$ and so that $I(A) \subset I(I(A))$. ■

Problem 71 *Let X be a non-empty set and let $I : 2^X \longrightarrow 2^X$ be an operator. Call $A \in 2^X$ I-open if $I(A) = A$. Show that if $I(X) = X$, $I(A) \subset A$, $I(A \cap B) = I(A) \cap I(B)$ and $I(I(A)) = I(A)$, then the collection of I-open sets forms a topology on X.*

To prove this, we first prove that $A \subset B \implies I(A) \subset I(B)$: $A = A \cap B \implies I(A) = I(A) \cap I(B) \subset I(B)$.

Solution 72 *Let $\tau = \{A \in 2^X : I(A) = A\}$. Then, $\emptyset \subset I(\emptyset)$ for any set in general and, conversely, $I(\emptyset) \subset \emptyset$. Thus, $\emptyset \in \tau$. Let $U_1, U_2 \in \tau$. Then, $I(U_1) = U_1$ and $I(U_2) = U_2$ so that $U_1 \cap U_2 = I(U_1) \cap I(U_2) = I(U_1 \cap U_2)$ by (c) and hence $U_1 \cap U_2 \in \tau$. Now let $n \in \mathbb{N}$ and $A' = \{1, 2, ..., n-1\}$ and $U_i, U_n \in \tau$ for $i \in A'$. Then, $U_i \in \tau \implies I\left(\bigcap_{i=A'}^{n} U_i\right) = \bigcap_{i=A'}^{n} U_i$ and $U_n \in \tau \implies I(U_n) = U_n$. Hence $\bigcap_{i=A'}^{n} U_i \cap U_n = I\left(\bigcap_{i=A'}^{n} U_i\right) \cap I(U_n) = I\left(\bigcap_{i=A'}^{n} U_i \cap U_n\right)$ again by (c) so that $\bigcap_{i=A'}^{n+1} U_i \in \tau$. Now, in order to show that $U_\alpha \in \tau \implies \bigcup_{\alpha = A} U_\alpha \in \tau$ for an arbitrary indexing set, that is, for $U_\alpha = I(U_\alpha)$, we must have $\bigcup_{\alpha = A} U_\alpha = I\left(\bigcup_{\alpha = A} U_\alpha\right)$. We must first show that $I(A \cup B) = A \cup B$. Clearly, $I(A \cup B) \subset A \cup B$ by hypothesis. By assumption, $A = I(A)$ and*

$B = I(B)$. *Hence,* $A \subset A \cup B$ *and* $B \subset A \cup B \implies I(A) = A \subset I(A \cup B)$ *and* $I(B) = B \subset I(A \cup B)$. *Thus,* $A \cup B \subset I(A \cup B)$.

Now, $I\left(\bigcup_{\alpha=A} U_\alpha\right) \subset \bigcup_{\alpha=A} U_\alpha$ *by default. Conversely, given that* $U_\alpha = I(U_\alpha)$, *for each* α, *we have that* $U_\alpha \subset \bigcup_{\alpha=A} U_\alpha \implies I(U_\alpha) = U_\alpha \subset I\left(\bigcup_{\alpha=A} U_\alpha\right)$. *Thus,* $\bigcup_{\alpha=A} U_\alpha \subset I\left(\bigcup_{\alpha=A} U_\alpha\right)$.

Note that the assumption $U = I(U)$ is crucial. For example, let $U_1 = \left[0, \frac{1}{2}\right]$ and $U_2 = \left[\frac{1}{2}, 1\right]$ under the usual topology of $\mathbb{R}$, then, $I(U_1 \cup U_2) = I([0,1]) = (0,1)$ but $I(U_1) \cup I(U_2) = \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \neq [0,1]$.

Problem 73 *Let* (X, τ) *be a topological space and let* $\{U_\alpha\}_{\alpha \in A}$ *be a collection of subsets. Such a collection is called* ***locally finite*** *if for every* $x \in X$, *there is a neighborhood* $x \in V$ *such that* $|\{\alpha \in A : V \cap U_\alpha \neq \emptyset\}| < \infty$. *In other words, there is a neighborhood of everypoint that intersects only finitely many sets in the collection. Show that if* $\{U_\alpha\}_{\alpha \in A}$ *is a locally finite collection of closed sets, then* $\bigcup_{\alpha \in A} U_\alpha$ *is a closed subset of* X.

Proof. To show that $\bigcup_{\alpha \in A} U_\alpha$ is closed, we will show that $\left(\bigcup_{\alpha \in A} U_\alpha\right)^c = \bigcap_{\alpha \in A} U_\alpha^c$ is open. Let $x \in \bigcap_{\alpha \in A} U_\alpha^c$. Then, $x \in U_\alpha^c \subset X$ for each α. Since $\{U_\alpha\}_{\alpha \in A}$ is locally finite, there exists an open set V such that $x \in V$ and $V \cap U_\alpha \neq \emptyset$ for finitely many α. Let $I = \{\alpha \in A : V \cap U_\alpha \neq \emptyset\}$. Then, $V \cap \bigcap_{\alpha \in A} U_\alpha^c = V \cap \bigcap_{i \in I} U_i^c = \bigcap_{i \in I} V \cap U_i^c$ is open. Note that $x \in V \cap \bigcap_{i \in I} U_i^c = U$ (say) so that U is a neighborhood of x. Clearly, $U \subset \bigcap_{i \in I} U_i^c$. Furthermore, $U \cap \bigcup_{\alpha \in A} U_\alpha = \emptyset$ since $U \subset U_i^c$ for each $i \in I$ so that $x \in \bigcap_{\alpha \in A} U_\alpha^c$. Hence for each $x \in \bigcap_{\alpha \in A} U_\alpha^c$, there exists U such that $x \in U \subset \bigcap_{\alpha \in A} U_\alpha^c$. Thus, $\bigcup_{\alpha \in A} U_\alpha$ is closed. ■

Problem 74 *Let* X *be a space with topologies* τ_1 *and* τ_2 *such that* $\tau_1 \subset \tau_2$. *For* $T \subset X$, *let* $Cl_1(T)$ *(resp.* $Cl_2(T)$*) denote the closure with respect to* τ_1 *(resp.* τ_2*). Prove that* $Cl_2(T) \subset Cl_1(T)$. *Formulate and prove a similar statement for interiors.*

Solution 75 *If* $T = \emptyset$, *then* $Cl_1(T) = \emptyset = T = Cl_2(T)$ *trivially. Thus, assume that* $T \neq \emptyset$. *Then, since* $T \subset Cl(T)$ *in general, we have that* $Cl_1(T) \neq \emptyset \neq$

$Cl_2(T)$. *Now let $\mathcal{A}_2$ be the collection of closed sets (relative to containing τ_2) containing T. Since $\tau_1 \subset \tau_2$, then for any closed set K in τ_1, K^c is open in $\tau_1 \implies K^c$ is open in $\tau_2 \implies K$ is closed in τ_2. Thus, $\mathcal{A}_1 \subset \mathcal{A}_2$, where $\mathcal{A}_1$ is the collection of closed sets (relative to τ_1) containing T from which we have* $Cl_2(T) = \bigcap_{A \in \mathcal{A}_2} A \subset \bigcap_{A \in \mathcal{A}_1} A = Cl_1(T)$.

The corresponding statement for interior operator is as follows: Let (X, τ_1) and (X, τ_2) be two topologies on X. If $\tau_1 \subset \tau_2$ and $U \subset X$, then $I_1(U) \subset I_2(U)$. Proof: Again, assume that $I_1(U) \neq \emptyset$, for otherwise the statement will be trivial. Let $U_\alpha \in \tau_1$ be open sets contained in U and let $\mathcal{U}_1 = \{U_\alpha : U_\alpha$ is open in τ_1 and $U_\alpha \subset U\}$. That is, $I_1(U) = \bigcup_{U_\alpha \in \mathcal{U}_1} U_\alpha$. $\tau_1 \subset \tau_2$ implies that $U_\alpha \in \tau_2$ so that $\mathcal{U}_1 \subset \mathcal{U}_2 = \{U_\alpha : U_\alpha$ is open in τ_2 and $U_\alpha \subset U\}$. Thus, $\bigcup_{U_\alpha \in \mathcal{U}_1} U_\alpha \subset \bigcup_{U_\alpha \in \mathcal{U}_2} U_\alpha$. That is, $I_1(B) \subset I_2(B)$.

2.3 Kuratowski Axioms

Just as we can define a topology using open sets, we can very well define a topology using closed sets. Such an approach was first propsed by Kazimierz Kuratowski by an operation $Cl : 2^X \longrightarrow 2^X$ where X is any non-empty set, satisfying the following:

1. $Cl(\emptyset) = \emptyset$
2. $A \subset Cl(A)$
3. $Cl(A \cup B) = Cl(A) \cup Cl(B)$
4. $Cl(Cl(A)) = Cl(A)$

Proposition 76 $A \subset B \implies Cl(A) \subset Cl(B)$

Proof. Let $A \subset B$. $B \cap A = A$. Also, $B = B \cap X = B \cap (A \cup A^c) = (B \cap A) \cup (B \cap A^c) = (B \cap A^c) \cup A = (B - A) \cup A$. By 3, $Cl(B) = Cl(B - A) \cup Cl(A)$. In particular, this means that $Cl(A) \subset Cl(B)$. ■

These are exactly the properties the operator $A \longmapsto \overline{A}$ satisfies. The operator Cl is called the Kuratowski Closure operator. Kuratowski called a set closed if $Cl(A) = A$. In this case, the collection $\tau = \left\{A \in 2^X : A^c = Cl(A^c)\right\}$ gives rise to a topology on a non-empty set X.

Proof. Since $Cl(\emptyset) = \emptyset$, thus $\emptyset^c = X \in \tau$. By definition of universal set $Cl(X) \subset X$. Converse holds by (2) above. Hence $Cl(X) = X$ so that $\emptyset = X^c \in \tau$ so that **T1** holds.

For **T2**, let $U_\alpha \in \tau$ where α is an index in an arbitrary indexing set A. Let $B_\alpha = U_\alpha^c$. Then, $\bigcap_{\alpha \in A} B_\alpha \subset Cl\left(\bigcap_{\alpha \in A} B_\alpha\right)$. To show the reverse inclusion, note that $\forall \gamma \in A$, $\bigcap_{\alpha \in A} B_\alpha \subset B_\gamma$ and that $Cl\left(\bigcap_{\alpha \in A} B_\alpha\right) \subset Cl\,(B_\gamma)$ by **Proposition 76**. From $B_\gamma = Cl\,(B_\gamma)$, we have $Cl\left(\bigcap_{\alpha \in A} B_\alpha\right) \subset B_\gamma$ for each γ so that $Cl\left(\bigcap_{\alpha \in A} B_\alpha\right) \subset \bigcap_{\alpha \in A} B_\alpha$. That is, $Cl\left(\bigcap_{\alpha \in A} B_\alpha\right) = \bigcap_{\alpha \in A} B_\alpha$ so that $\bigcap_{\alpha \in A} B_\alpha$ is closed. In other words, $\left(\bigcap_{\alpha \in A} B_\alpha\right)^c = \left(\bigcap_{\alpha \in A} U_\alpha^c\right)^c = \left(\left(\bigcup_{\alpha \in A} U_\alpha\right)^c\right)^c = \bigcup_{\alpha \in A} U_\alpha \in \tau$.

Finally, let $U_i \in \tau$ where i is in index in a finite indexing set I. Let $A_i = U_i^c$. Then, A_i is closed so that $Cl\,(A_i) = A_i$ and also that $Cl\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} Cl\,(A_i)$ by (3) above so that $Cl\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} A_i$ and hence $\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c = \bigcap_{i \in I} U_i \in \tau$ so that **T3** holds. ■

2.4 Limit Points

Definition 77 *Let (X, τ) be a topological space and let $A \subset X$ be non-empty. A point $x \in X$ is said to be a **limit point** of A if every neighborhood $U \ni x$ intersects A in a point other than x.*

In metric spaces, the definition takes the following form: $x_0 \in X$ of a set $A \subset X$ is a limit point if $\forall \epsilon > 0$, $N_\epsilon(x_0)$ contains points of A other than x_0. Notice that x_0 need not be a member of A. The set of all limit points of a set A is denoted by A^d. If a point is not a limit point, then it is called an **isolated point**.

Theorem 78 $\overline{A} = A \cup A^d$

Proof. Let $x \in A^d$. Then, every neighborhood of x intersects A in a point other than x. In particular, every neighborhood of x intersects A so $x \in \overline{A}$. That is, $A^d \subset \overline{A}$. Since $A \subset \overline{A}$, we have that $A^d \cup A \subset \overline{A}$. To show the converse, let $x \in \overline{A}$. Then, either $x \in A$ or $x \notin A$. If $x \in A$, then $x \in A \cup A^d$ and we are done. If $x \notin A$, then since $x \in \overline{A}$, every neighborhood of x intersects A. Since $x \notin A$, every neighborhood of x must intersect A at point(s) other than x, so that $x \in A^d$ by definition of A^d. Hence $x \in A \cup A^d$. In either case, $A^d \cup A \supset \overline{A}$. ■

Corollary 79 *Let (X, τ) be a topological space and let $A \subset X$ be non-empty. Then, A is closed $\iff$ $A^d \subset A$.*

Proof. A is closed $\iff \overline{A} = A \iff A = A \cup A^d \iff A^d \subset A$ ■

Problem 80 *Let (X, τ) be a T_1 topological space and let $A \subset X$. Show that if $x \in A^d \iff$ any neighborhood of x contains infinitely many points of A.*

Proof. Let x be a limit point of A. Then, by definition of limit point, every neighborhood of x intersects A at points other than x. Let U be such a neighborhood but assume that U contains finitely many points. That is, $U = \{u_1, ..., u_n\}$ with $x = u_k$ for some $k \in I = \{1, 2, ..., n\}$, then $U = \bigcup_{i \in I} \{u_i\}$. Since in a T_1 space, every singleton is closed, it follows that $\{u_i\}$ is closed for each i. Define $V = \bigcup_{i \in I,\ i \neq k} \{u_i\}$. Since finite union of closed sets is closed, thus V is closed $\implies V^c$ is open. Since $x \notin V$, we must have $x \in V^c$ so that V is a neighborhood of x. Also, by construction of V^c, we have excluded points of U which intersect with A. Thus, $V^c \cap (A \backslash \{x\}) = \emptyset$, implying the contradiction that there exists a nieghborhood of x which has an empty intersection with A at points other than A and that x is not a limit point. Thus the assumption that U is finite must be wrong and every neighborhood U of x contains infinitely many points.

Conversely, let $x \in X$ and assume that every neighborhood of x which intersects A at points other than x is infinite. In particular, this means that every neighborhood of x which intersects A at points other than x is non-empty. This is exactly the definition of a limit point and hence x is a limit point of A. ■

Problem 81 *Recall that the lower limit topology on $\mathbb{R}$ is the topology with basis $\mathcal{B} = \{[a, b) : a < b\}$. What is the closure of $\left(0, \sqrt{2}\right)$ in the lower limit topology?*

Solution 82 *The neighborhoods of any element are of the form $[a, b)$, $[b, \infty)$ or $\mathbb{R} = (-\infty, \infty)$. Let $x \in \mathbb{R}$. Then, $\mathbb{R} \cap \left[0, \sqrt{2}\right) \setminus \{x\} \neq \emptyset$, so we can skip $\mathbb{R}$. Let $[a, b)$ be a neighborhood of 0. Then, $a \leq 0 < b$ and $[a, b) \cap \left[0, \sqrt{2}\right) \setminus \{0\} \neq \emptyset$. If $[b, \infty)$ is a neighborhood of 0, then $[b, \infty) \cap \left[0, \sqrt{2}\right) \setminus \{0\}$. Hence $0 \in \left(0, \sqrt{2}\right)^d$. Next, we will show that no negative number is a limit point of $\left(0, \sqrt{2}\right)$. Let $\alpha < 0$. Consider the neighborhood $[\alpha, 0)$. $[\alpha, 0) \cap \left(0, \sqrt{2}\right) \setminus \{\alpha\} = \emptyset$. Hence no negative number is a limit point of the set $\left(0, \sqrt{2}\right)$. Furthermore, $\sqrt{2} \notin \left(0, \sqrt{2}\right)^d$ since the no element in the collection of neighborhoods $\left\{\left[\sqrt{2}, \sqrt{2} + p\right) : p \in \mathbb{Q}\right\}$ of $\sqrt{2}$ intersects $\left(0, \sqrt{2}\right)$ at any point, let alone $\sqrt{2}$. In particular, this also means that all real numbers$>$ $\sqrt{2}$ are also not limit points of $\left(0, \sqrt{2}\right)$. Hence $\left(0, \sqrt{2}\right)^d = \{0\}$ and so $Cl\left(\left(0, \sqrt{2}\right)\right) = \left[0, \sqrt{2}\right)$.*

Definition 83 *A subset M of a metric space X is said to be **dense** in X if $\bar{M} = X$ or X is said to be **separable** if it has a countable subset which is dense in X.*

More technically, if M is dense in X, then $\forall x_0 \in X$ and $\forall \epsilon > 0$, $B(x_0; \epsilon)$ will contain points of M.

This definition essentially says that we can have limit points of A within it and the result can equal to the parent set. One good example is the set of rationals and the real set. We all know that the set of rationals is not complete. In Analysis, real numbers can be constructed using Dedekind cuts or the addition of limits to every Cauchy sequence. That is, $\bar{\mathbb{Q}} = \mathbb{R}$ or that the set of rationals are dense in the set of reals. The complex plane, too, can be separated from the irrational real and imaginary parts against the rational ones.

Exercise 84 *The following conditions are equivalent:*

1. *M is dense in X*
2. *For every $x \in X$, there exists a sequence in M which converges in X.*
3. *Every nonempty open subset of X contains an element of M.*

Proposition 85 *A discrete metric space (X, d) is separable if and only if X is countable.*

Proof. Let $X = \{x_1, x_2, ..., x_n\}$ be a countable set and let $M = \{x_i, ..., x_j\}$ be a subset for $1 \leq i, j \leq n$. Then, for $d(x_i, x_j) < \epsilon$, we have $x_i = x_j$ if ϵ is less than 1. Hence, any open ball $B(x_k; \epsilon)$ for $1 \leq k \leq n$ will contain only the element x_k and no point is a limit point. Hence, no proper subset of X can have a limit point. Therefore, any M will not be dense in X. Since there are no limit points in X, $X^d = \emptyset$. We therefore have $\bar{X} = X \cup X^d = X$. Hence, X is dense in X and, therefore, separable.

Conversely, assume that X is separable, that is, $\exists M \subset X$ such that $\bar{M} = X$ but M^d is empty for the same reason as above. Therefore, $\bar{M} = M \cup M^d = X$ implies $M = X$ is the only possible subset. Since M (or X) does not have any limit points, every point is an isolated point. Hence, the set is countable. ■

Simply put, no proper subset of separable X can have limit points if the metric is discrete. It is surprising that we can have limit points in a specific metric space whenever we can count the elements of that space, and conversely.

3 Sequences

3.1 Sequences in Topological Spaces

Definition 86 *Let (X, τ) be a topological space. A **sequence** in X is a function $f : \mathbb{N} \longrightarrow X$.*

Denote $f(i) = x_i$. Then, $\{x_i : i \in \mathbb{N}\}$ is the range of the sequence, usually called the sequence itself. This is shortened to $\{x_i\}_{i=1}^{\infty}$ to indicate range.

Definition 87 *A sequence $\{x_i\}_{i=1}^{\infty}$ in a topological space (X, τ) is said to* ***converge to a point*** *x in X if for all neighborhoods $U \ni x$, $\exists N \in \mathbb{N}$ such that $x_i \in U$ for all $i \geq N$.*

In such a case, x is called a **limit of** $\{x_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ is said to converge to x. This is denoted by $\lim_{n \to \infty} x_n = x$ or $x_n \longrightarrow x$ as $n \longrightarrow \infty$.

Notice the use of the quantifier "a".

Example 88 *Let (X, τ) be a topological space and let $f(i) = a$ for some $a \in X$. Then, $\{x_i\}_{i=1}^{\infty} = \{a\}$, the constant sequence. In particular, $x_i \to a$.*

Limits are not necessarily unique. Consider

$$(X, \tau) = (\{a, b, c\}, \{\emptyset, X, \{a, b\}, \{b, c\}, \{b\}\})$$

Then, then for the sequence $f(i) = b$ for all i, $b \to a$, $b \to b$ and $b \to c$.

Such an unfortunate situation does not arise in Hausdorff spaces

Proof. Let $x_i \to a, b$ with $a \neq b$. Then, $\exists U, V$ open such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$. But by convergence, $\exists N \in \mathbb{N}$ such that $x_i \in U \cap V$ for all $i \geq N$, a contradiction. ∎

In the discrete topology, only the sequences which eventually become constant are convergent.

Proof. Let $x_i \to x$ with x_i not eventually constant. Then, for the open set $\{x\}$, there is no N such that $x_i \in \{x\}$ when $n \geq N$. ∎

In the indiscrete topology, every sequence converges to every point.

Theorem 89 *If X be a non-empty set equipped with two topologies τ, τ' with $\tau \subset \tau'$. If $x_i \to x$ in τ', then $x_i \to x$ in τ.*

Proof. Let $x_i \to x$ in τ' and let U be a neighborhood of x in τ. Then, $U \in \tau'$ so that $\exists N \in \mathbb{N}$ such that $x_i \in U$ for all $i \geq N$. Since U was arbitrary, therefore $x_i \to x$ in τ. ∎

The converse doesn't hold, though. This is because there are more open sets one needs to verify the condition of convergence for. As an example, consider the K-topology on $\mathbb{R}$ and the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. In $\mathbb{R}$, this sequence converges to 0 but this is not so in the K-topology: for $(-1, 1) \setminus K$, a neighborhood of 0, there exists no natural number $\frac{1}{n} \in (-1, 1) \setminus K$, in the first place.

In fact, this prompts us to observe that limit points of a set and a limit of a sequence are essentially different beasts in topological spaces. This is because the topology determines which sequence converges. Can we use the information on the limit points of a set to determine the topology? As one might expect, with a such a rhetorical question posed and because we've seen wilder in arbitrary topological spaces, the answer is that it is not necessary, unless the topology is first countable.

Definition 90 *Let (X, τ) be a topological space and let $A \subset X$. The* ***sequential closure*** *of A, denoted by $S(A)$, is the set of all points that are limits of sequences in A.*

In general, the following holds:

Theorem 91 $A \subset S(A) \subset \overline{A}$

Proof. Let $x \in A$. Then, the constant sequence x converges to x, hence $x \in S(A)$. Let $z \in S(A)$. Then, $\exists \{x_i\}_{i=1}^{\infty} \subset A$ such that $x_i \to z$. Let U be a neighborhood of z. Then, $\exists N \in \mathbb{N}$ such that $x_i \in U$ for $i \geq N$. Thus, $U \cap A \backslash \{z\} \neq \emptyset$ so that $x \in \overline{A}$. ∎

Proposition 92 *In the co-countable topology, for X uncountable and K a proper uncountable subset of X, a sequence converges $\iff$ the sequence is eventually constant.*

Proof. ($\implies$) Let $\{x_i\}_{i=1}^{\infty} = N \subset K$ be a convergent sequence and let $x \in S(N)$. Define $U := K \backslash \{x_n : x_n \neq x\}$. Then $U^c = K^c \cup \{x_i : n \in \mathbb{N} \wedge x_n = x\}$, which is countable. Hence U is open. Since $x_n \to x$, $\exists N \in \mathbb{N}$ such that $x_n \in U$ for $n \geq N$. By definition of U this means $x_n = x$ for all $n \geq N$. Thus, $\{x_i\}_{i=1}^{\infty}$ is eventually constant.

($\impliedby$) Holds trivially in any space. ∎

In this topology, if we regard $\{x_i\}_{i=1}^{\infty} = T$ as a set, then T is closed in τ. Furthermore, if $x_i \to x$ then $\{x_i\}_{i=1}^{\infty}$ is eventually constantly x. Thus, $S(T) = \overline{T}$.

Of course the converse doesn't hold in general but does so under what's called the first countable topology. In metric spaces, the situation is a lot more simpler, as we will see in the next section but let us first see what a first countable topology is.

Definition 93 *Let (X, τ) be a topology and let $x \in X$ and let $\mathcal{B}_x$ be a collection of neighborhoods of x. Then, $\mathcal{B}_x$ is called a* ***local basis*** *of x if, for $x \in U$ with U open, $\exists V \in \mathcal{B}_x$ such that $V \subset U$. That is, each neighborhood of x includes some member of $\mathcal{B}_x$. (X, τ) is called* ***first countable*** *if $\forall x$, $\mathcal{B}_x$ is countable. (X, τ) is called* ***second countable*** *if (X, τ) has a countable basis.*

Lemma 94 *Let (X, τ) be a topological space with basis $\mathcal{B}$ and let $x \in X$. Then, $\{U \in \mathcal{B} : x \in U\}$ is a local base at x.*

Proof. Let V be a neighborhood of x. Then, $\exists U_\alpha \in \mathcal{B}$ such that $V = \bigcup_{\alpha \in A} U_\alpha$ for all $\alpha \in A$, where A is some indexing set. In particular, $\exists i \in A$ such that $x \in U_i \subset V$. ∎

Theorem 95 *Let (X, τ) be a first countable topological space. Then, $S(A) = \overline{A}$.*

Proof. We only need to prove that $\overline{A} \subset S(A)$. Let $x \in \overline{A}$ and let $\{U_i : i \in \mathbb{N}\}$ be a local base at x.

We can assume, WLOG, that $U_i \subset U_j$ for $i \geq j$. If not then we can let $V_1 = U_1$, $V_2 = U_1 \cap U_2$, ..., $V_n = \bigcap_{i=1}^{n} U_i$. Then, $\{V_i : i \in \mathbb{N}\}$ is a local base at x because for $U \ni x$, $\exists k$ such that $U_k \subset U$ and $V_k \subset U_k \subset U$.

Now, $\forall i$, $U_i \cap A \neq \emptyset$. Let $x_i \in U_i \cap A$. Then, $x_i \to x$.

To see this, let $U \ni x$ be a neighborhood of x. Then, $\exists k$ such that $U_k \subset U$. But $U_i \subset U_k$ for $i \geq k$. Thus, $x_i \in U_k \subset U$ for $i \geq k$.

Thus, $x \in S(A)$. ■

Limits of subsequences are, however, the same as those for the parent sequence.

Problem 96 *Let (X, τ) be a topological space and let $\{x_i\}_{i=1}^{\infty}$ be a sequence in X. Prove that $x_i \to x \iff$ every subsequence x_{i_j} of x_i converges to x.*

Solution 97 ($\Longrightarrow$) *First, we note that $i_j \geq j$ for all i.*
Proof. *Since i_j is a subsequence of natural numbers j, it follows $i_1 \geq 1$ and, by induction, assume that $i_j \geq j$. Then, $i_{j+1} > i_j \geq j \implies i_{j+1} \geq j+1$.* ■

Now, let $x_i \longrightarrow x$. Then, $\forall U \in \tau$ with $x \in U$, there exists $N \in \mathbb{N}$ such that $x_i \in U$ $\forall i \geq N$. Now, if $j > N$, then $i_j \geq N$. Thus $x_{i_j} \in U$ for $i_j \geq N$. Thus the subsequence x_{i_j} converges to x. Since x_{i_j} was arbitrary, therefore any subsequence $x_{i_j} \longrightarrow x$.

($\Longleftarrow$) *Assume that every subsequence x_{i_j} of x converges to x. Then, since x_i is a subsequence of itself, it follows that x_i converges to x.*

3.2 Sequences in Metric Spaces

Definition 98 *A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) is said to* ***converge*** *or* ***to be convergent*** *if there is an $x \in X$ such that $\lim\limits_{n\to\infty} d(x_n, x) = 0$. Alternatively, a sequence is called convergent if $\forall \epsilon > 0$, $\exists N$ such that $d(x_n, x) < \epsilon$ whenever $n > N$.*

If we cannot find an N for any given ϵ, or that if the sequence fails to be convergent, we say that this sequence diverges. A sequence $\{x_n\}_{n=1}^{\infty}$ is bounded if its range $\{x_n\}_{n=1}^{\infty}$ is bounded.

In metric spaces, limits are unique. This is because a metric space is a particular instance of a Hausdorff Space.
Proof. Let $\lim\limits_{n\to\infty} x_n = l_1$ and $\lim\limits_{n\to\infty} x_n = l_2$ be two limits. Then, $\forall \epsilon > 0$, we can find N_1 and N_2 such that $d(x_n, l_1) < \epsilon/2$ and $d(x_n, l_2) < \epsilon/2$ for $n > N_1, N_2$. Let $N = \max(N_1, N_2)$. Then, $d(l_1, l_2) < d(x_n, l_1) + d(x_n, l_2) < \epsilon/2 + \epsilon/2 = \epsilon$ whenever $n > N$, $\forall \epsilon > 0$. The condition $d(l_1, l_2) < \epsilon$ implies $l_1 = l_2$ ■

Exercise 99 *If a sequence converges to a point, then any subsequence will converge to that point.*

Proposition 100 *If a sequence converges, then it is bounded.*

Proof. Let $x_n \to x$. Then, we can be assured that we will definitely have a (very large) natural number N such that $d(x_n, x) < \epsilon$ $\forall n > N$ and $\forall \epsilon > 0$. Let $m = \max(d(x_1, x), d(x_2, x), ..., d(x_N, x), \epsilon)$. Then, $d(x_n, x) < m$ $\forall n$ which really means that every element of the sequence is bounded. ■

The converse, however, is usually false. Consider the series $\frac{(-1)^n}{2}$. This series is bounded by ± 1 yet does not converge as it keeps on alternating between $-1/2$ and $1/2$.

Corollary 101 *If a sequence is unbounded, then it is divergent.*

Proposition 102 *If x is a limit point of a subset A of a metric space (X, d), then there exists a sequence such that $x_n \to x$.*

Proof. This is a particular instance of **Problem 80** because metric spaces are T_1. To see to the exact proof, we need to construct a sequence that converges to this limit point. Since x is a limit point, then we can rest assured that we have an open ball centred at x of ϵ radius contained in X and containing points other than x, by definition. Hence, we can collect such points and call them x_n. Therefore, $d(x_n, x) < \epsilon$. What we need now is to prove that we have an N such that this is valid for $n > N$. Since epsilon was arbitrary, we can let it depend on the index n. So, $\epsilon = 1/n$, say. From a collection of the natural numbers, we will always have an N such that $N\epsilon < 1$. This can be seeing by applying the Archimedean property of real numbers. Now, we have $1 > N\epsilon$ or $1 > N/n$ or $n > N$, which establishes the proof. ■

Theorem 103 *Let M be a nonempty subset of a metric space (X, d). Then*

1. $x \in \bar{M} \iff \exists x_n \in M$ such that $x_n \to x$
2. M is closed $\iff x_n \in M$ such that $x_n \to x$ implies that $x \in M$.

Proof. For bullet 1, we've proven that any limit point will have a sequence convergent to it. The converse is a trivial result of the definition of convergence and limit points. Bullet two follows by observing that if M is closed, then $M = \bar{M}$ ■

Proposition 104 *If $x_n \longrightarrow x$ and $y_n \longrightarrow y$ in X, then $d(x_n, y_n) \longrightarrow d(x, y)$.*

Proof. $\forall \epsilon > 0$, we can find N_1 and N_2 such that

$$d(x_n, x) < \epsilon/2$$

and

$$d(y_n, y) < \epsilon/2$$

for $n > N_1, N_2$. Let $N = \max(N_1, N_2)$. Then,

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x) + d(x, y) + d(y_n, y) \\ &\Longrightarrow d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y_n, y) \end{aligned}$$

Also,

$$\begin{aligned} d(x, y) &\leq d(x, x_n) + d(x_n, y_n) + d(y_n, y) \\ &\Longrightarrow -[d(x, x_n) + d(y_n, y)] \leq d(x_n, y_n) - d(x, y) \end{aligned}$$

Using the two inequalities, we have

$$|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y) < \epsilon/2 + \epsilon/2 = \epsilon$$

i.e. $|d(x_n, y_n) - d(x, y)| < \epsilon \ \forall n > N$. ■

Notice that in this proof, we've treated $d(x_n, y_n)$ as a sequence with the index n. Thus, the metric function is continuous.

4 Continuous Functions

4.1 Continuous Functions in Metric Spaces

Open sets also play a role in connection with continuous mappings, where continuity is a natural generalisation of the continuity known from calculus and is defined as follows:

Definition 105 *Let (X, d_1) and (Y, d_2) be metric spaces. A mapping*

$$T : X \longrightarrow Y$$

is said to be ***continuous*** *at a point $x_0 \in X$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$d_2(T(x), T(x_0)) < \epsilon$$

whenever $d_1(x, x_0) < \delta$.

T is said to be continuous if it is continuous at every point of X. Alternatively, this definition could be phrased as follows:

Theorem 106 *A mapping T of a metric space X into a metric space Y is continuous if and only if the inverse image of any open subset of Y is an open subset of X.*

Proof. Let $B \subset Y$ be an open set and let $T^{-1}(B) = \{x \mid T(x) \in B\}$. We need to prove that $T^{-1}(B)$ is open. Let $T(x_0) \in B$. Since $T(x_0)$ is an interior point, we have $d_2(T(x), T(x_0)) < \epsilon \ \forall \epsilon > 0$. Since T is continuous, this ensures the existence of a δ such that $d_1(x, x_0) < \delta$. Hence for any ϵ or for any open set, we can find a δ or an open set $A(x_0; \delta) = \{x \mid d_1(x, x_0) < \delta\}$. Hence the inverse image of every open set is open.

The converse of the proof is trivial. We start with $B(T(x_0); \epsilon)$, an open set, such that $T^{-1}(B) = A$ is open by suggesting that this set satisfies $d(x, x_0) < \delta$ for every $x \in A$, guaranteeing the existence of the required δ. ■

Theorem 107 *Let (X, d_1) and (Y, d_2) be metric spaces. A mapping*

$$T : X \longrightarrow Y$$

is continuous at a point $x \in X$ $\iff$

$$x_n \longrightarrow x \implies T(x_n) \longrightarrow T(x)$$

Proof. If T is continuous at x, then for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_2(T(y), T(x)) < \epsilon$ whenever $d_1(y, x) < \delta$. If $x_n \longrightarrow x$, then we can label the points y when we have an $n > N$ so that $d_1(x_n, x) < \delta$, which is possible when $d_2(T(x_n), T(x)) < \epsilon$, for every $\epsilon > 0$ and $n > N$.

The converse of the proof is trivial. ■

Definition 108 *Let (X, d_X) and (Y, d_Y) be metric space and let $f : X \longrightarrow Y$. Then, f is **uniformly continuous** if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $d_X(a, b) < \delta \implies d_Y(f(a), f(b)) < \epsilon$ for every $a, b \in X$.*

It is easy to see that uniform continuity implies continuity. The converse is not true. Consider $f(x) = x^2$ on $\mathbb{R}$ with standard metric. This is continuous but not uniformly continuous. Let $\epsilon = 1$ and $\delta > 0$. Let $x = y - \frac{\delta}{2}$. Then, $|x - y| = \frac{\delta}{2} < \delta$. Then, $|f(x) - f(y)| = \left|y^2 - x^2\right| = |y + x| \, |y - x| \leq |y + x| \, \delta = \left|2y\delta - \delta^2\right|$. For $y >> 0$, then $\left|2y\delta - \delta^2\right| > 1$

Of particular interest is the convergence of sequence of functions f_n. However, in this case, we also have to consider the domain of the functions, as well. In the ordinary notion of continuity, this convergence will depend on each point of the domain, giving the name "point-wise convergence". Apart from this notion of continuity, we also have the notion of uniform continuity, in which the elements of the domain do not matter. Thus, in uniform continuity, we have

Definition 109 *A sequence of functions $f_n(x)$ **converges uniformly** if $\forall \epsilon > 0$, $\exists N$ such that $d(f_n(x), f(x)) < \epsilon \ \forall x$ whenever $n > N$*

In uniform convergence, we have convergence of functions for every element of the domain. This type of convergence is important when dealing with spaces involving continuous functions. In fact, if a sequence of function converges to a function, that is $\lim_{n \to \infty} f_n = f$, then this is valid for all x. That is, $\lim_{n \to \infty} f_n(x) = f(x)$. That ϵ in the definition will depend upon x if the convergence is point-wise and will not if the convergence is uniform.

There is another notion of convergence in the space of functions:

Definition 110 *A sequence of functions $f_n(x)$ **converges pointwise** if $\forall \epsilon > 0$ $\forall x$, $\exists N$ such that $d(f_n(x), f(x)) < \epsilon$ whenever $n > N$.*

The difference is subtle: here N depends both on x and ϵ whereas in the former, for each ϵ you need to be able to find an N for all x in the domain of the function. In other words, N can depend on ϵ but not on x. Like uniform and ordinary continuity, the former definition is global in nature whereas the other talks about convergence depending on the domain.

Exercise 111 *Show the uniform convergence implies pointwise convergence but not conversely.*

Theorem 112 *If a series of functions converges uniformly, then the limit is continuous*

Proof. Let $f_n(x) \longrightarrow f(x)$ uniformly. Then, we have an N such that $\forall \epsilon > 0\, \forall x$, $d(f_n(x), f(x)) < \frac{\epsilon}{3}$ whenever $n > N$. We also have continuous $f_n(x)$ so that $\forall \epsilon > 0$, $\exists \delta$ such that $d(f_n(x), f_n(y)) < \frac{\epsilon}{3}$ whenever $d(x, y) < \delta$. The uniform continuity is valid for all $x \in \mathcal{D}(f)$ and, in particular, whenever $d(x, y) < \delta$. Hence, whenever $d(x, y) < \delta$, we have

$$\begin{aligned} d(f(x), f(y)) &\leq d(f_n(x), f(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

so that $f(x)$ is continuous. ■

Note that the above proof has to be valid for all x. Hence, convergence in $C[a, b]$ is always uniform and never point-wise.

4.2 Continuous Functions in Topological Spaces

Continuous functions are topology preserving maps. That is, let (X, τ_X) and (Y, τ_Y) be two topological spaces. Let $f : X \longrightarrow Y$ be a function. To figure out an apropriate class of functions, we may ask if $\tau = \{f(U) : U \in \tau_X\}$ forms a topology?. No, since not every f is surjective and hence $Y \in \tau_2$ may not necessarily be true. However, the collection $\tau = \left\{f^{-1}(U) : U \in \tau_Y\right\}$ works. To successfully show this, we recall two facts from set-theory: $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$ and $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$ for any function f.

Theorem 113 $\tau = \left\{f^{-1}(U) : U \in \tau_Y\right\}$ *is a topology on* X*.*

Proof. $\emptyset = f^{-1}(\emptyset)$ and $X = f^{-1}(Y)$. Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of sets in τ for an arbitrary indexing set A. Then, there exists $U'_\alpha \in \tau$ such that $U_\alpha = f^{-1}(U'_\alpha)$ and so $f^{-1}\left(\bigcup_{\alpha \in A} U'_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(U'_\alpha) = \bigcup_{\alpha \in A} U_\alpha$. Since $\bigcup_{\alpha \in A} U'_\alpha \in \tau_Y$, we must have that $\bigcup_{\alpha \in A} U_\alpha \in \tau$. Now let $\{U_i\}_{i \in I}$ be a finite collection of sets in τ. Then, there exists $U'_i \in \tau$ such that $U_i = f^{-1}(U'_i)$ and so $f^{-1}\left(\bigcap_{i \in I} U'_i\right) = \bigcap_{i \in I} f^{-1}(U'_i) = \bigcap_{i \in I} U_i$. Since $\bigcap_{i \in I} U'_i \in \tau_Y$, we must have that $\bigcap_{i \in I} U_i \in \tau$. ■

This topology has a name: $f^*(\tau_Y)$, the pull-back topology.

Definition 114 *Let* (X, τ_X) *and* (Y, τ_Y) *be two topological spaces and let* $f : X \longrightarrow Y$ *be a function. Then,* f *is* ***continuous*** $\forall U \in \tau_Y$, $f^{-1}(U) \in \tau_X$*. That is, if every open set in* Y *has an open pre-image.*

Example 115 *Let* (X, τ_X) *and* (Y, τ_Y) *be two topological spaces and let* $f : X \longrightarrow Y$ *be a function. Then, the constant function* $f(x) = c$ *is continuous: let* U *be an open set in* τ_Y*. If* $U = \emptyset$*, then* $f^{-1}(\emptyset) = \emptyset$ *implies* $f^{-1}(U)$ *is open.*

Let $U \neq \emptyset$. Then, $c \in U$ or $c \notin U$. If $c \notin U$, then $f^{-1}(U) = \emptyset$ so that $f^{-1}(U)$ is open. If $c \in U$, then, $f^{-1}(U) = X$ by definition of constant function and hence $f^{-1}(U)$ is again open.

Example 116 *If (X, τ_X) is a discrete topological space and (Y, τ_Y), is any topological space, then $f : X \longrightarrow Y$ is continuous: let U be an open set in τ_Y. If $U = \emptyset$, then $f^{-1}(\emptyset) = \emptyset$ implies $f^{-1}(U)$ is open. If $U \neq \emptyset$ but if $f^{-1}(U) = \emptyset$ (f may not be surjective), then again $f^{-1}(U)$ is open. Assume $f^{-1}(U) \neq \emptyset$ for $U \neq \emptyset$. Then, $U = \{y : f(x) = y \text{ and } x \in X\}$, by definition of a function. Then, we let $A = \{x : f(x) = y \in U\}$, so that A is a non-empty subset of X. By construction, $f^{-1}(U) = A$. Since τ_X is discrete and A is a non-empty subset of X, it follows that A is open. That is, $f^{-1}(U) = A$ is open.*

Example 117 *If (X, τ_X) is any topological space and (Y, τ_Y), is the indiscrete topological space, then $f : X \longrightarrow Y$ is continuous: let U be an open set in τ_Y. Then either $U = \emptyset$ or $U = Y$. If $U = \emptyset$, then $f^{-1}(\emptyset) = \emptyset$ implies $f^{-1}(U)$ is open. If $U = Y$, then $f^{-1}(Y) = X$ is open.*

Proposition 118 *f is continuous iff $f^*(\tau_Y) \subset \tau_X$ and $f^*(\tau_Y)$ is the coarsest topology on X which makes f continuous.*

Proof. ($\Longrightarrow$) Let f is continuous. As proved above, $f^*(\tau_Y)$ is a topology on X. If $U \in f^*(\tau_Y)$, then U is the pre-image of an open set by definition of pull-back topology, so that $U \in \tau_X$, by definition of continuity. Assume that there exists a topology $\tau \subset f^*(\tau_Y)$. Then, there exists an open set U in $f^*(\tau_Y)$ and not open in τ but then U is the pre-image of an open in set in τ_Y so that U is open in τ, a contradiction.

($\Longleftarrow$) Let $f^*(\tau_Y) \subset \tau_X$ and $f^*(\tau_Y)$ is the coarsest topology on X. Assume that f is not continuous. Then, there exists an open map such that $f^{-1}(U) = V$ (say) is not open. But this would imply that V is not open in τ_X so that V is not open in $f^*(\tau_Y)$, a contradiction. ■

Theorem 119 *Let (X, τ_X) and (Y, τ_Y) be two topological spaces and let $f : X \longrightarrow Y$ be a function. Let B be a basis (resp. subbase) for τ_Y. Then, f is continuous $\Longleftrightarrow$ $f^{-1}(U) \in \tau_X$ for every $U \in B$.*

Proof. ($\Longrightarrow$) Let $U \in B$. Then, U is open and so, if f is continuous, then $f^{-1}(U)$ is open.

($\Longleftarrow$) Let $U \subset V$ be open in Y. Then. $U = \bigcup U_\alpha$ for $U_\alpha \in B$. Then, $f^{-1}(U) = f^{-1}\left(\bigcup U_\alpha\right) = \bigcup f^{-1}(U_\alpha)$, which is open. ■

We can show that the ϵ-δ definition in calculus implies the definition for topological space. To recall, $f : \mathbb{R} \longrightarrow \mathbb{R}$ be ϵ-δ continuous $\Longleftrightarrow$ $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Let U be an open set, ι be a point of U. Since each point of U is an interior point, we can have $\epsilon_\iota > 0$ such that $\iota \in N_{\epsilon_\iota}(\iota) \subset U$ where $N_{\epsilon_\iota}(\iota) = \{\zeta : |\iota - \zeta| < \epsilon_\iota\}$. We will first show that $U = \bigcup_{\iota \in U} N_{\epsilon_\iota}(\iota)$. Clearly, $N_{\epsilon_\iota}(\iota) \subset U \implies \bigcup_{\iota \in U} N_{\epsilon_\iota}(\iota) \subset \bigcup_{\iota \in U} U = U$. Conversely, let $y \in U$. Since U is open and every point of U is an interior point, then, $y \in N_{\epsilon_\iota}(\iota) \subset I$ for some $\iota \in U$ so that $y \in \bigcup_{\iota \in U} N_{\epsilon_\iota}(\iota)$ and thus $U \subset \bigcup_{\iota \in U} N_{\epsilon_\iota}(\iota)$.

To show that $f^{-1}(U)$ is open, let $a \in f^{-1}(U)$. That is, $f(a) \in U = \bigcup_{\iota \in U} N_{\epsilon_\iota}(\iota)$. Then, there exists ϵ_ι such that $f(a) \in N_{\epsilon_\iota}(\iota)$ or that $a \in f^{-1}(N_{\epsilon_\iota}(\iota))$. We now need to show that $f^{-1}(N_{\epsilon_\iota}(\iota))$ is open. Let $\epsilon = \epsilon_\iota - |f(a) - \iota|$. By definition of ϵ-δ continuity, we are guaranteed to find a δ such that $|x-a| < \delta \implies |f(x) - f(a)| < \epsilon$. Consider $x \in N_\delta(a)$. Then, $|f(x) - \iota| = |f(x) - f(a) + f(a) - \iota| \leq |f(x) - f(a)| + |f(a) - \iota| < \epsilon + |f(a) - \iota| = \epsilon_\iota$ so that $f(x) \in N_{\epsilon_\iota}(\iota) \subset U$, or that $x \in f^{-1}(N_{\epsilon_\iota}(\iota)) \subset f^{-1}(U)$. Thus, $N_\delta(a) \subset f^{-1}(N_{\epsilon_\iota}(\iota))$. Effectively, this means that for any $a \in f^{-1}(N_{\epsilon_\iota}(\iota))$, we have a δ such that $N_\delta(a) \subset f^{-1}(N_{\epsilon_\iota}(\iota))$, implying that a is an interior point of $f^{-1}(N_{\epsilon_\iota}(\iota))$. Since a was arbitrary, we have that $f^{-1}(N_{\epsilon_\iota}(\iota))$ is open. In summary, we've shown that for each open ball contained in U, the inverse image of that open ball is open. Since the arbitrary union of open sets is open and $U = \bigcup_{\iota \in U} N_{\epsilon_\iota}(\iota)$, therefore $f^{-1}(U) = f^{-1}\left(\bigcup_{\iota \in U} N_{\epsilon_\iota}(\iota)\right) = \bigcup_{\iota \in U} f^{-1}(N_{\epsilon_\iota}(\iota))$ is open.

Definition 120 *Let (X, τ_X) and (Y, τ_Y) be two topological spaces and let $f : X \longrightarrow Y$ be a function. Then, f is* ***continuous at*** *$x \in X$ if for every neighborhood V which contains $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$.*

Theorem 121 *Let (X, τ_X) and (Y, τ_Y) be two topological spaces and let $f : X \longrightarrow Y$ be a function. Then, f is continuous $\iff$ f is continuous at every $x \in X$*

Proof. ($\implies$) Let f be continuous, $x \in X$ and let V be a neighborhood of $f(x)$. Then, by continuity, $U = f^{-1}(V)$ is open, $U \ni x$ and $f(U) \subseteq V$

($\impliedby$) Let $V \subseteq Y$ be open. For each $x \in f^{-1}(V)$, we know that there will be an open set $U_x \ni x$ such that $f(U_x) \subseteq V$. Let $U = \bigcup_{x \in f^{-1}(V)} U_x$. Then, U is open is it is the union of open sets. We need to show that $U = f^{-1}(V)$. Let $x \in f^{-1}(V)$. Then, $x \in U_x$ so that $x \in \bigcup_{x \in f^{-1}(V)} U_x = U$ and hence $f^{-1}(V) \subset U$. On the other hand, if $y \in U$, $\exists U_x$ such that $y \in U_x$. By hypothesis, $f(U_x) \subset V$ so that $y \in f^{-1}(V)$. ■

Theorem 122 *Let (X, τ_X), (Y, τ_Y) and (Z, τ_Z) be topological spaces and let $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ be continuous functions. Then, $g \circ f$ is continuous.*

Proof. Let U be an open set in τ_Z. Then, $g^{-1}(U)$ is open in τ_Y since g is continuous. Since f is continuous, $f^{-1}\left(g^{-1}(U)\right)$ is open in τ_X. Since

$(g \circ f)^{-1}(U) = f^{-1}\left(g^{-1}(U)\right)$, then for any open set U in τ_Z, $(g \circ f)^{-1}(U)$ is open in τ_X. Hence $g \circ f$ is continuous. ∎

The converse is not true. Let $X = Y = Z = \mathbb{R}$ with the usual topology

$$f(x) = \begin{cases} 1 & x \in \mathbb{R} \backslash \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$$

and

$$g(x) = \begin{cases} 0 & x \in \mathbb{R} \backslash \mathbb{Q} \\ 1 & x \in \mathbb{Q} \end{cases}$$

Then, $(g \circ f)(x) = 1$ for all x. That is, a constant function, which, as we've seen in the previous question, is continuous. However, f and g are both not continuous. Proof: Let $U = (a, b)$ be any open set in $\mathbb{R}$ with $a, b \in \mathbb{R}$ and $a < b$. Then, U contains both rationals and irrationals and so, $f^{-1}(U) = g^{-1}(U) = \{0, 1\}$, which is not an open set because either point is not an interior point: we can never find an open set contained in $\{0, 1\}$, centered at 0 or 1 for, if we did, then we would get the contradiction that $\{0, 1\}$ is uncountable!

Theorem 123 *Let (X, τ_X) and (Y, τ_Y) be two topological spaces and let $f : X \longrightarrow Y$ be a function. Then, f is continuous $\Longleftrightarrow$ the inverse image of every closed set is closed.*

Proof. ($\Longrightarrow$) Let $C \subset Y$ be closed. Then, C^c is open, so if f is continuous, then $f^{-1}(C^c) = \left[f^{-1}(C)\right]^c$ and so $f^{-1}(C)$ is closed.

($\Longleftarrow$) Let U be an open set in Y. Then, $f^{-1}(U^c)$ is closed in X or that $f^{-1}(U)$ is open. ∎

Problem 124 *Let (X, τ_X) and (Y, τ_Y) be two topological spaces and let $f : X \longrightarrow Y$ be a function. Prove that f is continuous $\Longleftrightarrow$ for each $B \subset Y$, $\overline{f^{-1}(B)} \subset f^{-1}\left(\overline{B}\right)$*

In other words, f is continuous if and only if the closure of the preimage of B is contained in the preimage of the closure of B

Solution 125 ($\Longrightarrow$) *Let f be continuous and let $B \subset Y$. Then, $\overline{B}$ is closed and so $Y - \overline{B}$ is open. Thus, $f^{-1}\left(Y - \overline{B}\right)$ is open. Next, we prove that $f^{-1}\left(Y - \overline{B}\right) = X - f^{-1}\left(\overline{B}\right)$. $f^{-1}\left(Y - \overline{B}\right) = \left\{x : f(x) \in Y - \overline{B}\right\} = \left\{x : f(x) \notin \overline{B}\right\} = X - \left\{x : f(x) \in \overline{B}\right\} = X - f^{-1}\left(\overline{B}\right)$. Now, since $Y - \overline{B}$ is open and f is continuous, then $X - f^{-1}\left(\overline{B}\right)$ is open and, therefore, $f^{-1}\left(\overline{B}\right)$ is closed so that $f^{-1}\left(\overline{B}\right) = \overline{f^{-1}\left(\overline{B}\right)}$. Now, $B \subset \overline{B}$ implies $f^{-1}(B) \subset f^{-1}\left(\overline{B}\right)$ by properties of a function, which implies $\overline{f^{-1}(B)} \subset \overline{f^{-1}\left(\overline{B}\right)}$ by properties of closure operator. Hence $\overline{f^{-1}(B)} \subset f^{-1}\left(\overline{B}\right) = \overline{f^{-1}\left(\overline{B}\right)}$*

($\Longleftarrow$) *Let $\overline{f^{-1}(B)} \subset f^{-1}\left(\overline{B}\right)$ for some arbitrary $B \subset Y$. Let A be an arbitrary closed set in Y. We need to show that $f^{-1}(A)$ is closed. Using $\overline{f^{-1}(B)} \subset f^{-1}\left(\overline{B}\right)$ and $A = \overline{A}$, we have $\overline{f^{-1}(A)} \subset f^{-1}(A)$. That is, $f^{-1}(A)$ contains all its limit points. Hence $f^{-1}(A)$ is closed.*

How do sequences behave under continuous functions?

Theorem 126 *Let (X, τ_X) and (Y, τ_Y) be two topological spaces and let $f : X \longrightarrow Y$ be a continuous function. If $\{x_i\}$ is a sequence in X and $x_i \longrightarrow a$ for some $a \in X$, then $\{f(x_i)\}$ in Y converges to $f(a)$*

Proof. Let V be a neighborhood of $f(a)$. Then, $U = f^{-1}(V)$ is open and $a \in U$. Then, $\exists N \in \mathbb{N}$ such that $x_i \in U$ for $i \geq N \implies f(x_i) \in V$ for $i \geq N$ so that $f(x_i)$ converges to $f(a)$. ∎

Does the converse hold? Let (X, τ_X) and (Y, τ_Y) be two topological spaces and let $f : X \longrightarrow Y$ be a function such that if $\{x_i\} \subset X$ and $x_i \longrightarrow a \implies f(x_i) \longrightarrow f(a)$. Does f have to be continuous? No. Let X be any uncountable set and $Y = X$. Let τ_X be the co-countable topology and let τ_Y be discrete. Define $id_A : X \longrightarrow Y$. Then, id_A is not continuous because $f^*(\tau_Y)$ is also discrete topology but clearly, $\tau_X \subset \tau_Y$ strictly. Also, $x_i \longrightarrow x$ in $\tau_Y \iff \{x_i\}_{i=1}^{\infty}$ is eventually constantly x, unless we add the requirement that X is first countable.

4.3 Profinite topology

Let $X = \mathbb{Z}$. We can define a topology on $\mathbb{Z}$ using arithmetic progressions in $\mathbb{Z}$ and the empty set as a basis. That is, for $N_a(b) = \{a + nb : n \in \mathbb{Z}\}$ and $B = \{N_a(b) : a, b \in \mathbb{Z}\} \cup \{\emptyset\}$. Let $x \in \mathbb{Z}$. Then, $x \in N_0(x)$ because $x = 0 + 1x$ where $a = 0$ and $b = x$. This satisfies **B1**. Next, let $N_{a_1}(b_1)$, $N_{a_2}(b_2) \in B$ and $x \in N_{a_1}(b_1) \cap N_{a_2}(b_2)$. Then, $x = a_1 + nb_1$ and $x = a_2 + mb_2$ for some $n, m \in \mathbb{Z}$, by definition of elements of B. Notice that $a_1 + nb_1 = a_2 + mb_2 \implies nb_1 - mb_2 = a_2 - a_1$ forms a Diophantine Equation. The solution of this Diophantine equation, that is, numbers n, m, exist, by assumption $(x \in N_{a_1}(b_1) \cap N_{a_2}(b_2))$. Thus, if there is one element in the intersection of two elements of B, (which we assume there is), there are others. Let $d = \gcd(b_1, b_2)$. Then, $n - k\frac{b_2}{d}$ and $m + k\frac{b_1}{d}$ are other solutions of the Diophantine equation $nb_1 - mb_2 = c$ where $c = a_2 - a_1$ with unknowns m, n, where k is an integer. Thus, we are guaranteed to find intersection points y of $N_{a_1}(b_1)$ and $N_{a_2}(b_2)$ at $y = a_1 + \left(n - k\frac{b_2}{d}\right) b_1 = a_1 + nb_1 - k \operatorname{lcm}(b_1, b_2)$ and $y = a_2 + \left(m + k\frac{b_1}{d}\right) b_2 = a_2 + mb_2 + k \operatorname{lcm}(b_1, b_2)$, provided that $N_{a_1}(b_1) \cap N_{a_2}(b_2) \neq \emptyset$. Thus, for $b = \operatorname{lcm}(b_1, b_2)$ and $a = a_2 + mb_2$ or $a_1 + nb_1$ (both work!), the set $N_a(b) \subset N_{a_1}(b_1) \cap N_{a_2}(b_2)$ and $x \in N_a(b)$. This shows that B is a basis.

The topology generated by this basis is certainly coarser than the discrete topology but only slightly so.

Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}/3\mathbb{Z}$. Let $\mathbb{Z}/3\mathbb{Z}$ have the discrete topology. Then, basis for $\mathbb{Z}/3\mathbb{Z} = \mathbb{B} = \{\{[0]\}, \{[1]\}, \{[2]\}\}$. Then, $f^{-1}(\{[0]\})$ is the coset $3\mathbb{Z}$, an arithmetic progression, and is open in the profinite topology, $f^{-1}(\{[1]\}) = 1 + 3\mathbb{Z}$, open in profinite topology and lastly $f^{-1}(\{[2]\}) = 2 + 3\mathbb{Z}$, open in profinite topology.

In fact, let G be any group with discrete topology. Let $f : \mathbb{Z} \longrightarrow G$ be a homomorphism. Then, f is continuous if $\mathbb{Z}$ has the profinite topology. If $g \in G$.

case I, $g \notin f(\mathbb{Z})$, then $f^{-1}(g) = \emptyset$. If $g \in f(\mathbb{Z})$, then $f^{-1}(g) = h + \ker f$ in general $\{g\}$ has a pre-image of $N_h(\ker f)$, which is open.

Thus another way of defining profinite topology on $\mathbb{Z}$ is by saying that it is the coarsest topology on $\mathbb{Z}$ such that all homomorphisms to finite groups are continuous.

5 Construction of Topologies 2

5.1 Subspace Topology

Definition 127 *Let (X, τ) be a topological space and let $A \subset X$. Can we put a topology on A? Yes. Let $\tau_A = \left\{U \in 2^A : U = A \cap V \text{ for some } V \in \tau\right\}$. Such a topology is called the* ***subspace topology****.*

Theorem 128 *τ_A is a topology on A*

Proof. $A = A \cap X$ so A is open in τ_A. Next, $\emptyset = A \cap \emptyset$ so that $\emptyset \in \tau_A$. Next, let $\{U_\beta\}_{\beta \in B}$ be a collection of open sets in τ_A. Then, for each $\beta \in B$, $\exists U_\beta' = A \cap U_\beta'$. Then, $\bigcup\limits_\beta U_\beta = \bigcup\limits_\beta \left(A \cap U_\beta'\right) = \left(\bigcup\limits_\beta A\right) \cap \left(\bigcup\limits_\beta U_\beta'\right)$. The first term is A, the second is open in X. Hence arbitrary union of open sets is open in τ_A. Finally, let U_1, U_2 be two open sets in τ_A. Then, there exists $U_1', U_2' \in \tau$ such that $U_i = A \cap U_i'$ for $i = 1, 2$. Then, $U_1 \cap U_2 = A \cap U_1' \cap A \cap U_2' = A \cap \left(U_1' \cap U_2'\right)$. Thus, $U_1 \cap U_2$ is open in τ_A. ■

Theorem 129 *Let (X, τ) be a topological space, let $A \subset X$ and τ_A be the subspace topology. Let $i_A : A \longrightarrow X$ be the inclusion map. Then, i_A is continous. Morevoer, the subspace topology is the coarsest topology on A so that i_A is continuous*

Proof. Let $U \subset X$ be open. Then, $i_A^{-1}(U) = A \cap U$ and the pre-image of U is open in A. This topology is coarsest because of it forms the pull-back topology. More specifically, let τ' be a topology on A for which i_A is continuous. To prove the 2nd part, we will prove the contrapositive that if there is a coarser topology than the subspace topology, then i is not continuous. Since there is a topology τ' coarser than τ_A, there must be an open set V be an open set of in τ such that $A \cap V$ is not open in τ'. But then $i^{-1}(V) = A \cap V$ is not open, implying that i is not continuous. ■

Example 130 *Let $X = \mathbb{R}$ with usual topology and $A = \mathbb{Z}$. Then, the subspace topology is the discrete topology: let $U \in 2^{\mathbb{Z}}$. Then, $U = \mathbb{Z} \cap (\inf U - 1, \sup U + 1)$ where $(\inf U - 1, \sup U + 1) \subset \mathbb{R}$ is open and hence U is open in the subspace topology.*

Example 131 *Let* $X = \mathbb{R}^2$ *and* $A = \{(x,0) : x \in (a,b) \subset \mathbb{R}\} \subset X$. *There is an order on* $\mathbb{R}^2$ *such that* $(x', y') \leq (x, y)$ *if* $x \geq x'$ *or if* $x = x'$ *and* $y \geq y'$. *Define open sets on* $\mathbb{R}^2$ *as follows: let* $p, q \in \mathbb{R}$. *Then, interval* $(p,q) = \{r \in \mathbb{R}^2 : p < r < q\}$. *This is exactly how the* $\mathbb{R}^1$ *Euclidean space may be constructed, as well. Open intervals are then strips in which lower line of the left-most strip is excluded and upper line of the right-most strip is excluded. Let* $q = (x, 1)$ *and* $p = (x, -1)$ *be points in* $\mathbb{R}^2$ *and* $B = (x, 0) \in A$ *be a set. Then,* $B \cap (p,q) = (a,b)$.

Lemma 132 *Let* (X, τ_X) *and* (Y, τ_Y) *be topological spaces and suppose that* $X = A \cup B$ *where* A *and* B *are both open (or closed) in* X. *Let* $f : A \longrightarrow Y$ *and* $g : B \longrightarrow Y$ *be continous functions such that if* $x \in A \cap B$, *then* $f(x) = g(x)$. *Then, there exists unique continuous function* $h : X \longrightarrow Y$ *given by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

Proof. Let $A, B \in \tau_X$ and let $V \subset Y$ be open. Let $U_A = f^{-1}(V)$ and $U_B = g^{-1}(V)$. By hypothesis, U_A is open in τ_A and U_B is open in τ_B so that there exists open sets U'_A, U'_B in ambient space such that $U_A = A \cap U'_A$ and $U_B = B \cap U'_B$. This step is crucial, as can be seen from the counter-example below. Since $A \cup B = X$ and each are open in τ_X, we must have

$$\begin{aligned} U_A \cup U_B &= (A \cap U'_A) \cup (B \cap U'_B) \\ &= ((A \cap U'_A) \cup B) \cap ((A \cap U'_A) \cup U'_B) \\ &= (A \cup B) \cap (U'_A \cup B) \cap (A \cup U'_B) \cap (U'_A \cup U'_B) \\ &= X \cap (U'_A \cup B) \cap (A \cup U'_B) \cap (U'_A \cup U'_B) \\ &= (U'_A \cup B) \cap (A \cup U'_B) \cap (U'_A \cup U'_B) \end{aligned}$$

open. Let $U = U_A \cup U_B$. We now show that $h^{-1}(V) = U$. This is because if $x \in U$, then, WLOG, assume $x \in U_A$, $h(x) = f(x) \in V$ so that $U \subset h^{-1}(V)$. Similarly, if $x \in h^{-1}(V)$, then WLOG assume $x \in A \implies h(x) = f(x) \in V \implies x \in U_A \implies U = h^{-1}(V)$. Since V was arbitrary, we are done. ■

Example 133 $X = \mathbb{R}$, $A = (-\infty, 0]$ *and* $B = [0, \infty)$. *Let* $h : \mathbb{R} \longrightarrow \mathbb{R}$ *such that*

$$h(x) = \begin{cases} x & \text{if } x < 0 \\ 2x & \text{if } x \geq 0 \end{cases}$$

if $f : A \longrightarrow \mathbb{R}$ *and* $g : B \longrightarrow \mathbb{R}$ *as* $f = h|_A$, $g = h|_B$. *Then,* $f^{-1}((a,b)) = (a,b)$ *if* $b \leq 0$, $f^{-1}((a,0]) = (a,0]$, $g^{-1}((a,b)) = \left(\frac{a}{2}, \frac{b}{2}\right)$, $a \geq 0$ *and* $g([0,b)) = \left[0, \frac{b}{2}\right)$.

As a counter example, consider the indicator function of rationals and irrationals. Then, $f(x) = 1$ for $A = \mathbb{Q}$, $g(x) = 0$ for $B = \mathbb{Q}^c$. As shown in **Example 115**, constant functions are always continuous, considering the subspace topologies. However, the function $h(x)$, which is the indicator function for rational numbers, is not continuous. This is because both sets A and B are neither closed nor open in the Euclidean topology.

5.2 Homeomorphisms

Definition 134 *Let (X, τ_X) and (Y, τ_Y) be two topological spaces and let $f : X \longrightarrow Y$ be a continuous function. Then, f is a **homoemorphism** if f is bijective and if f^{-1} is continuous*

Example 135 *$X = (-\pi/2, -\pi/2)$ and $Y = \mathbb{R}$ with the usual topologies. For $x \in X$, define $f(x) = \tan x$. Then, f satisfies properties above.*

As a non-example, consider $X = [0, 2\pi)$ and $Y = \left\{(x, y) : x^2 + y^2 = 1\right\}$ taking the subspace topologies. For $t \in X$, define $f(t) = (\cos t, \sin t)$. It is easy to see that f is continuous and bijective but f^{-1} is not continuous: for $\epsilon > 0$, $f([0, \epsilon))$ is not open in Y.

5.3 Quotient Topology

Can we get a natural topology from a map? Let A, X be sets. If $f : A \longrightarrow X$ is an ijnection, then $f(A) \subset X$ and we can get a topology. This is dual to quotient topology:

Definition 136 *Let $p : X \longrightarrow A$ be a surjection and let (X, τ) be a topological space. The set $\tau_p = \left\{U \in 2^A : p^{-1}(U) \in \tau\right\}$ is called the **quotient topology**.*

This is a topology where a set is open if its pre-image is open in the ambient space.

Theorem 137 *τ_p is a topology on A*

Proof. We can get a topology on A even if p is not surjective. Since $p^{-1}(\emptyset) = \emptyset$ for any function in general and p in particular, and $\emptyset$ is open in any topology, therefore $\emptyset \in \tau_p$. Next, as a set, $p^{-1}(A) = \{x : p(x) \in A\}$. Since p is a function, $p(x) \in A$ for any $x \in X$. Thus, $p^{-1}(A) = X$. Since X is open in X, thus $A \in \tau_p$. Let Γ be an arbitrary indexing set, γ be an index and let $U_\gamma \in \tau_p$ for each γ. Then, $p^{-1}(U_\gamma)$ is open in X. Since X is a topological space, it must be that $\bigcup_{\gamma \in \Gamma} p^{-1}(U_\gamma)$ is open in X. That is, $\bigcup_{\gamma \in \Gamma} p^{-1}(U_\gamma) = p^{-1}\left(\bigcup_{\gamma \in \Gamma} U_\gamma\right)$ is open in X. Hence $\bigcup_{\gamma \in \Gamma} U_\gamma \in \tau_p$. Finally, let I be a finite indexing set, i be an index and let $U_i \in \tau_p$ for each γ. Then, $p^{-1}(U_i)$ is open in X. Since X is a topological space, it must be that $\bigcap_{i \in I} p^{-1}(U_\gamma)$ is open in X. That is, $\bigcap_{i \in I} p^{-1}(U_\gamma) = p^{-1}\left(\bigcap_{i \in I} U_\gamma\right)$ is open in X. Hence $\bigcap_{i \in I} U_\gamma \in \tau_p$, showing that τ_p is a topology. ■

Note again that nowhere have we used the fact that p is surjective.

Definition 138 *Let (X,τ_X) and (A,τ_A) be topological spaces and $p : X \longrightarrow A$ be surjective map. p is said to be an* ***open map*** *(resp.* ***closed map****) if $U \in \tau_X \implies p(U) \in \tau_A$ (respectively, U is closed in $\tau_X \implies p(U)$ is closed in τ_A). p is a* ***quotient map*** *if $p^{-1}(U) \in \tau_X \iff U \in \tau_A$.*

Note that p is not assumed to be continuous.

Theorem 139 *Let (X,τ_X) be topological space and $p : X \longrightarrow A$ be surjective map. Then,*

1. *p is continuous if A is equipped with the quotient topology.*
2. *The quotient topology is the finest topology on A for which p is continuous.*
3. *Quotient topology is the unique topology for which p is a quotient map.*

Proof. **1.** Let $U \in \tau_p$. Then, $p^{-1}(U) \in \tau_X$ and so p is continuous. **2.** Let τ' be a topology on A for which p is continous and let $U \in \tau'$. Then, p is continuous $\implies p^{-1}(U) \in \tau_X$ and so $U \in \tau_p$ so that $\tau' \subset \tau_p$. **3.** Already proved by showing that it is the finest. ■

If it is not already clear that p has to be surjective, here is an alternate explaination: let $p : X \longrightarrow A$ be any map and let $B = p(X)$. Then, the subspace topology coming from regarding $B \subset A$ with the topology $\tau_p = \left\{U \in 2^A : p^{-1}(U) \in \tau\right\}$ on A is the same as the topology $\tau' = \left\{U \in 2^B : p^{-1}(U) \text{ is open in } X\right\}$ for the map $p : X \longrightarrow B$ but is strictly coarser than the topology τ_p.
Proof. Let $\tau_B = \{B \cap U : U \text{ is open in } A\}$. We need to show that $\tau_B = \tau' = \left\{U \in 2^B : p^{-1}(U) \text{ is open in } X\right\}$. Let $V \in \tau_B$. Then, $\exists$ an open set U in A such that $V = B \cap U$. Since U is open in A, $p^{-1}(U)$ is open in X. Then, $p^{-1}(V) = p^{-1}(B \cap U) = p^{-1}(B) \cap p^{-1}(U) = X \cap p^{-1}(U)$ so that $p^{-1}(V)$ is open in X. Hence $V \in \tau'$ so that $\tau_B \subset \tau'$. Conversely, let $U \in \tau'$. Then, $p^{-1}(U)$ is open in X so that $U \in \tau_p$ and that U is open in A. Furthermore, since X is a topology, $p^{-1}(U) \cap X$ is open in X or that $p^{-1}(U) \cap p^{-1}(B)$ is open in X. That is, $p^{-1}(B \cap U)$ is open in X. Thus, there exists an open set U in A such that $B \cap U$ is open in B. In other words, $U \in \tau_B$, showing that $\tau' \subset \tau_B$, concluding that $\tau_B = \tau'$.

The topology τ' is strictly coarser than τ_p. This is because $B \subset A \implies 2^B \subset 2^A$. Let $U \in \tau' \subset 2^B \subset 2^A$. Then, $p^{-1}(U)$ is open in X so that $U \in \tau_p$ and that U is open in A. It follows that $\left\{U \in 2^B : p^{-1}(U) \text{ is open in } X\right\} \subset \left\{U \in 2^A : p^{-1}(U) \text{ is open in } X\right\}$. That is, $\tau' \subset \tau_p$ or that τ_p is finer than $\tau\prime$. From $\tau_B = \tau'$, the open sets U of τ_p not in τ' are those for which $B \subset U$ but that $B \neq U$. ■

Thus, the condition of surjectivity in a quotient map gives tells us that the quotient topology is the finest topology on A for which p is continuous. Thus, In light of **1** in **Theorem 139**, some authors call the quotient map "stronger continuity".

Example 140 *Let $X = [0,1]$ and $Y = S^1 = \left\{(x,y) : x^2 + y^1 = 1\right\} \subset \mathbb{R}^2$. Let $f : X \longrightarrow Y$ such that $f(t) = (\cos 2\pi t, \sin 2\pi t)$. Then, any set open in the* *subspace topology on Y is open in the quotient topology. Thus, f is a quotient map. If $X = [0,1)$, then f is continuous so that subspace topology is coarser than quotient topology. However, f is not a quotient map because the map of the interval $[0,\epsilon)$ is not open so that quotient topology is not coarser than subspace topology.*

Example 141 (Identification spaces) *Let (X,τ) be a topological space, $f : X \longrightarrow Y$ be a surjection and Y have the quotient topology. Define $x \sim y$ if $f(x) = f(y)$. Let X^* be the set of equivalence classes. Then, $\pi : X \longrightarrow X^*$ such that $\pi(x) = [x]$ so that π is a surjection. There exists an $\widetilde{f} : X^* \longrightarrow Y$ such that $\widetilde{f} \circ \pi = f$. $\widetilde{f}$ is a bijection. If X^* has the quotient topology, then $\widetilde{f}$ is a homeomorphism.*

Problem 142 *Consider the equivalence relation on $\mathbb{R}$ given by $x \sim y$ if there is $\lambda > 0$ so that $y = \lambda x$.*

1. *How many equivalence classes are there? Describe them.*
2. *Let Y be the set of equivalence classes of this equivalence relation and let $p : \mathbb{R} \longrightarrow Y$ given by $p(x) = [x]$, where $[x]$ is the equivalence class containing x. Show that the quotient topology on Y is not Hausdorff.*
3. *Show that there is no Hausdorff topology on Y for which the map $p : \mathbb{R} \longrightarrow Y$ is continuous.*

Solution 143 *(1) There are 3 classes. 0 is not related to any other non-zero element. Let $x \neq 0$. Then, either $x > 0$ or $x < 0$. For the sake of contradiction, assume $x \sim 0$. Then, $\exists\lambda > 0$ such that $0 = x\lambda$. But $x > 0$, $\lambda > 0$ implies $x\lambda > 0$ so that $0 > 0$, a contradiction. In the second case, i.e., $x < 0$, if $x \sim 0$, then, $\exists\lambda > 0$ such that $0 = x\lambda$. But $x < 0$, $\lambda > 0$ implies $x\lambda < 0$ so that $0 < 0$, another contradiction. Hence $x \nsim 0$ for any $x \neq 0$. In other words, one class is $[0] = \{0\}$. The second class consists of positive real numbers: let $x > 0$ and $y > 0$. Then, $y = \frac{y}{x}x$. Clearly, $\frac{y}{x} > 0$ so that we can have $\lambda = \frac{y}{x}$ so that $x \sim y$. Thus, any two positive real numbers belong to one class so that the second class is $[x] = \{x : x \in \mathbb{R}$ and $x > 0\}$. The third class is of negative numbers: let $x < 0$ and $y < 0$. Then, $y = \frac{y}{x}x$. Clearly, $\frac{y}{x} > 0$ so that we can have $\lambda = \frac{y}{x}$. Thus, any two negative real numbers belong to one class so that the third class is $[x] = \{x : x \in \mathbb{R}$ and $x < 0\}$. Assume that there is a fourth class, say $[\alpha]$. Then, the Trichotomy law applied on the representative α gives $\alpha > 0$, $\alpha < 0$ or $\alpha = 0$. In either case, α belongs to either existing classes so that $[\alpha]$ is not different from already existing classes.*

(2) Denote $[+] = \{x : x \in \mathbb{R}$ and $x > 0\}$ and $[-] = \{x : x \in \mathbb{R}$ and $x < 0\}$. The quotient topology on $Y = \{[+], [-], [0]\}$ has to be the finest topology for which p is continuous. Clearly, $\{[0]\}$ cannot be an open in Y since $p^{-1}(\{[0]\}) = \{0\}$, which is not open. Neither can $\{[-], [0]\}$ and $\{[+], [0]\}$ since the inverse images of these sets are half-open intervals which are closed in the usual topology. Then, either $[-]$ has to be closed or $[+]$. Hence the topologies that remain are

$\{\emptyset, Y\}$ *or* $\{\emptyset, Y, \{[+]\}\}$, $\{\emptyset, Y, \{[-]\}\}$ *and* $\{\emptyset, Y, \{[+]\}, \{[-]\}, \{[+], [-]\}\}$. *The finest of these is the last one, which is not Hausdorff since the points* $[0]$ *and* $[+]$ *do not have any disjoint neighborhoods.*

(3) Already answered above.

5.4 Product Topology

We now move on to talk about product topologies but first we need to talk about Cartesian products for arbitrary sets.

Definition 144 *Let A be an indexing set and let X_α be a collection of sets for $\alpha \in A$. Let $X = \bigcup_{\alpha \in A} X_\alpha$. The* ***Cartesian Product*** *of $\{X_\alpha\}_{\alpha \in A}$, denoted by $\prod_{\alpha \in A} X_\alpha$, is the set of functions $f : A \longrightarrow X$ such that $f(\alpha) \in X_\alpha$.*

Thus, $\prod_{\alpha \in A} X_\alpha \subset X^A$.

Example 145 *For $A = \{1, 2\}$, the definition reduces to $X_1 \times X_2$. For $X_\alpha = \mathbb{R}$ for each α and $A = \{1, ..., n\}$. Then, $\prod_{\alpha \in A} X_\alpha = \mathbb{R}^n$. Similarly, if $X_\alpha = B$ for each α, then $\prod_{\alpha \in A} X_\alpha = B^A$.*

Definition 146 *A* ***projection function****, for each $\beta \in A$, is a map $\pi_\beta : \prod_{\alpha \in A} X_\alpha \longrightarrow X_\beta$, defined as $\pi_\beta(f) = f(\beta) \in X_\beta$.*

This map surjective.

Proof. Let $a \in X_\beta$. Then, by Axiom of Choice, $\exists f : A \longrightarrow X$, defined by $f(\beta) = a$ if $a \in X_\beta$. Then, $\exists f$ such that $\pi_\beta(f) = f(\beta) = a$. ■

Let (X_α, τ_α) be a topological space for each α. Let

$$B = \left\{ \prod_{\alpha \in A} U_\alpha : U_\alpha \in \tau_\alpha \text{ for each } \alpha \right\} \subset \prod_{\alpha \in A} X_\alpha$$

This is a basis. The operations on this are the usual, sensical ones: let U,$V \in B$. Then, $U = \prod_{\alpha \in A} U_\alpha$ and $V = \prod_{\alpha \in A} V_\alpha$ and $U \cap V = \prod_{\alpha \in A} (U_\alpha \cap V_\alpha) \in B$. Et cetera.

The topology on $\prod_{\alpha \in A} X_\alpha$ generated by B is called the **box topology**. For $A = \{1, 2\}$ and $X_\alpha = \mathbb{R}$, the box topology $\mathbb{R}^2$ is the same as the Euclidean space but things aren't necessarily this neat.

For example, let $A = \mathbb{N}$ and $X_\alpha = \mathbb{R}$. Then, $\prod_{\alpha \in A} X_\alpha = \mathbb{R}^{\mathbb{N}}$ is simply sequences of real numbers. Let $\mathbb{R}^{\mathbb{N}}$ have the box topology. Define $f : \mathbb{R} \longrightarrow \mathbb{R}^{\mathbb{N}}$

by constant sequences $f(t) = (t, t, ...)$. This function is not continuous: let $U = (-1,1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times ....$ This is an open set in the box topology. However, $f^{-1}(U) = \left\{r : r \in \left(-\frac{1}{n}, \frac{1}{n}\right) \forall n\right\} = \{0\}$ which is not open.

Problem here is the this box topology has too many open sets. To see this, let $f : Y \longrightarrow \prod_{\alpha \in A} X_\alpha$ and $f_\beta : Y \longrightarrow X_\beta$ with $\pi_\beta \circ f = f_\beta$ where $\pi_\beta : \prod_{\alpha \in A} X_\alpha \longrightarrow X_\beta$. Take $Y = \mathbb{R}$ and and $\prod_{\alpha \in A} X_\alpha = \mathbb{R}^{\mathbb{N}}$. Then, the $f_k(t) = t$ for each natural number k so that the we get "extra" open sets when we apply f. This box topology is then replaced by what's called the product topology: for a topological space (X_α, τ_α) be for each $\alpha \in A$, let

$$B = \left\{ \prod_{\alpha \in A} U_\alpha : U_\alpha \in \tau_\alpha \text{ for each } \alpha \text{ and } U_\beta = X_\beta \text{ for all but finitely many } \beta \in A \right\} \subset \prod_{\alpha \in A} X_\alpha$$

This is a basis for product topology.

Definition 147 *Let $\{X_\alpha, \tau_\alpha\}_{\alpha \in A}$ be a collection of topological spaces and $X = \bigcup_{\alpha \in A} X_\alpha$. Then, $\prod_{\alpha \in A} X_\alpha = \left\{f \in X^A : f(\alpha) \in X_\alpha\right\}$ has the* ***product topology*** *with basis B defined above.*

Proposition 148 *If $|A| < \infty$, then product topology = box topology.*

Example 149 *For $A = \{1, 2\}$ and $X_\alpha = \mathbb{R}$, the box topology $\mathbb{R}^2$ is the same as the Euclidean space. Also, for $U = (-1,1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times ...$ is not an open set if $A = \mathbb{N}$.*

Problem 150 *Let X be a topological space and let $\Delta \subset X \times X$ be the diagonal. In other words, $\Delta = \{(y, y) : y \in X\}$. Prove that X is Hausdorff if and only if Δ is a closed subset of $X \times X$.*

Solution 151 $(\Longrightarrow)$ *Let X be a Hausdorff space. Then, for each $y \in X$, $\{y\}^c$ is open. It follows that $\{y\}^c \times \{y\}^c = \{(y, y)\}^c$ is open in the product topology, and that $\bigcup_{y \in X} \{(y, y)\}^c = \Delta^c$ is open in the product topology, since arbitrary union of open sets is open. That is, Δ is closed.*

$(\Longleftarrow)$ *Let Δ be closed in $X \times X$. Then, Δ^c is open. Let $x, y \in X$ be two distinct points. Then, $(x, y) \notin \Delta$ or that $(x, y) \in \Delta^c$. Since Δ^c is open, (x, y) is an interior point so that there exists a neighborhood N in $X \times X$ such that $(x, y) \in N \subset \Delta^c$. Since N is an open set, $\exists$ basis elements U, V, open sets in X, with $x \in U$, $y \in V$, such that $N \supset U \times V$. Since $U \times V \subset \Delta^c$, it must be that $U \cap V = \emptyset$ for otherwise if $z \in U \cap V$, then $z \in U$ and $z \in V$ so that $(z, z) \in U \times V$ but that $U \times V \subset \Delta^c$ means that $(z, z) \in \Delta^c$, a contradiction.*

Theorem 152 *Let $\{X_\alpha, \tau_\alpha\}_{\alpha \in A}$ be a collection of topological spaces such that $\prod_{\alpha \in A} X_\alpha$ is the product topology. Let (Y, τ) be a topological space and let $f : Y \longrightarrow$*

$\prod_{\alpha \in A} X_\alpha$ *be a function. Then,* f *is continuous* $\iff$ $f_\beta = \pi_\beta \circ f$ *is continuous for each* $\beta \in A$*.*

Proof. ($\implies$) Assume that f is continuous. Let $\beta \in A$. To show that $f_\beta = \pi_\beta \circ f$ is continuous, let $U \subset X_\beta$ be open. Then, $f_\beta^{-1}(U) = f^{-1} \circ \pi_\beta^{-1}(U)$. We note that $\pi_\beta^{-1}(U) = \prod_{\alpha \in A} U_\alpha$ where $U_\alpha = U$ if $\alpha = \beta$ and $U_\alpha = X$ otherwise. But f is continuous so that $f^{-1} \circ \pi_\beta^{-1}(U)$ is open in Y. That is, projection maps are continuous. Note: this part holds even for box topology.

($\impliedby$) Let B be a basis for $\prod_{\alpha \in A} X_\alpha$ and let $U \in B$. Then, $\exists$ finite set $F \subset A$ such that $U = \prod_{\alpha \in A} U_\alpha$ where $U_\alpha \subset X_\alpha$ is open and $U_\alpha = X_\alpha$ if $\alpha \notin F$. For each $\alpha \in F$, let $V_\alpha = f_\alpha^{-1}(U_\alpha) \subset Y$ and let $V = \bigcap_{\alpha \in F} V_\alpha$ (is open in Y).

Now we show that $f^{-1}(U) = V$. Even though this is true without F being finite, but in this case, V may not be open. This is where the finiteness of F comes in.

$\forall \alpha \in F$, define $W_\alpha = \prod_{\beta \in A} S_\beta$ where

$$S_\beta = \begin{cases} U_\beta & \text{if } \beta = \alpha \\ X_\beta & \text{if } \beta \neq \alpha \end{cases}$$

We can check that $U = \bigcap_{\alpha \in F} W_\alpha$. Then, $f^{-1}(U) = f^{-1}\left(\bigcap_{\alpha \in F} W_\alpha\right) = \bigcap_{\alpha \in F} f^{-1}(W_\alpha) = \bigcap_{\alpha \in F} f_\alpha^{-1}(U_\alpha) = f_\alpha^{-1}\left(\bigcap_{\alpha \in F} U_\alpha\right) = V$. ∎

Example 153 $f : [0,1] \longrightarrow \mathbb{R}^2$ *with* $f(t) = (\cos t, \sin t)$ *and* $\pi_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *and* $\pi_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$*. Then,* $f_1(t) = \cos t = (\pi_1 \circ f)(t)$ *and* $f_2(t) = \sin t = (\pi_2 \circ f)(t)$*. This map is continuous in the product topology and, therefore, the box topology.*

6 Connectedness

Recall the famous

Theorem 154 (Intermediate Value Theorem) *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be continuous. Let* $a, b \in \mathbb{R}$ *such that* $f(a) > d$, $f(b) < d$*, then there exists* $c \in \mathbb{R}$ *such that* $f(c) = d$*.*

Can we say the same for a continuous $f : X \subset \mathbb{R} \longrightarrow \mathbb{R}$? That is, if $a, b \in X$ such that $f(a) > d$, $f(b) < d$, then does there exists $c \in X$ such that $f(c) = d$?

It may be true with $X=\mathbb{R}$, obviously. It is also true in $[0,1]$, $[0,1)$ or may be even the half-open infinite interval. It is obviously false for $X=(-2,1)\cup(1,2)$ for $f(x)=x^3$. The IVT is true when X is "one piece". Topologically, this is defined as follows:

Definition 155 *Let (X,τ) be a topological space. A **separation** of X is a pair (U,V) of open, non-empty disjoint sets such that $X=U\cup V$. X is said to be **connected** if it has no separations. If $A\subset X$ is a connected subspace if A is connected in the subspace topology. A set that is not connected is called disonnected.*

Thus, we are led to propose that the IVT works when the codomain, $\mathbb{R}$ in the above case, is connected. This is tied with completeness property of reals. To show this, we have a long way to go.

Example 156 *Consider $A=(-2,-1)\cup[1,2)\subset\mathbb{R}$. Then A is disconnected in the subspace topology with $U=(-2,-1)$ and $V=[1,2)$*

Example 157 *Let X be a set such that $|X|>1$ with discrete topology. Then, X is disconnected because $U=\{a\}$ and $V=\{a\}^c$ is a separation of X for $a\in X$.*

Example 158 *Let (X,τ) be a topological space and $A=\{a\}$ be a subset. Then, A is connected in the subspace topology.*

Example 159 *Let (X,τ) be the indiscrete topological space. Then, X is connected because there is no separation of X.*

Example 160 *For the usual topology on $\mathbb{R}$, the $\mathbb{Q}$ is disconected. Let r be any irrational number and let $U=\mathbb{Q}\cap(-\infty,r)$ and $V=\mathbb{Q}\cap(r,\infty)$. Then, U,V are open in the subspace topology and $U\cup V=\mathbb{Q}$. Also, if $Y\subset\mathbb{Q}$ with $|Y|>1$, then Y is disconnected if, for $a,b\in Y$ and irrational r such that $a<r<b$, then $U=Y\cap(-\infty,r)$ and $V=Y\cap(r,\infty)$ is a separation of Y.*

Example 161 *p-adic metric topology is disconnected because open balls are also closed*

Proposition 162 *Let (X,τ) be a topological space with $|X|>1$ and let $a\in X$. If $\{a\}$ is the only connected set for all a, then X is disconnected.*

Proof. If $|X|=2$, then $U=\{a\},V=\{b\}$ are connected and X is not. For arbitrary X, let U,V be open subsets of X such that $X=U\cup V$. If $|U|,|V|>1$, then, U,V are not connected, by hypothesis. Let (U_1,U_2) be a separation of U and (V_1,V_2) be a separation of V. Then, $U_1\cap U_2=\emptyset=V_1\cap V_2$ and so $U_i\cap V_j=\emptyset$ **(how?)** so that $U\cap V=\emptyset$. ■

Such sets are called **totally disconnected**.

Theorem 163 *X is connected $\Longleftrightarrow$ the only, non-empty open and closed subset of X is X itself.*

Proof. ($\Longleftarrow$) Let X be the only, non-empty open and closed subset of X. Suppose that (U,V) is a separation of X. Then, since $X = U \cup V$ and $U \cap V = \emptyset \implies U = V^c$ so U is also closed, the contradiction.

($\Longrightarrow$) If W is a proper, non-empty, open and closed subset of X, then $\left(W, W^C\right)$ is a seperation. ■

Theorem 164 *Let (X,τ) be a topological space with seperation (U,V). If $A \subset U$, then $\overline{A} \cap V = \emptyset$. In particular, V contains no limit points of A*

Proof. $\overline{A} \cap V \neq \emptyset \Longrightarrow V$ is a nieghborhood for some $x \in A \Longrightarrow V \cap A \neq \emptyset$ so that $U \cap V \neq \emptyset$. ■

Theorem 165 *If (C,D) is a seperation of X and $Y \subset X$ is connected, then $Y \subset C$ or $Y \subset D$.*

Proof. Suppose that $Y \cap C \neq Y \neq Y \cap D$. Let $U = Y \cap C$ and $V = Y \cap D$. Then, since C, D are open in X, V, U are open in Y. Since $C \cap D = \emptyset$ and $C \cup D = X$, then $U \cap V = \emptyset$. Furthermore, $U \cup V = (Y \cap C) \cup (Y \cap D)$

$= ((Y \cap C) \cup Y) \cap ((Y \cap C) \cup D)$

$= ((Y \cup Y) \cap (C \cup Y)) \cap ((Y \cup D) \cap (C \cup D))$

$= (Y \cap (C \cup Y)) \cap ((Y \cup D) \cap X)$

$= Y \cap (C \cup Y) \cap (Y \cup D)$

$= Y$, a contradiction. ■

Theorem 166 *Let (X,τ) be a connected topological space, (Y, τ_Y) be a topological space and $f : X \longrightarrow Y$ be a continuous map. Then, $f(X) \subset Y$ is connected.*

Proof. Let $D = f(X)$. Suppose that (U,V) be a separation of D. Then, $D = U \cup V$ so that $U' = f^{-1}(U)$ and $V' = f^{-1}(V)$ are open, since f is continous, so that $U' \cup V' = X$. These are non-empty since U and V are non-empty since $U, V \subset f(X)$. Now, suppose that $x \in U' \cap V'$, then, $f(x) \in U$ and $f(x) \in V$, implying the contradiction that U and V are not disjoint. Hence U' and V' are disjoint, so that (U', V') is separation of X, a contradiction. ■

Theorem 167 *Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of connected subsets of a topological space (X,τ). If $\bigcap_{\alpha \in A} U_\alpha \neq \emptyset$, then $\bigcup_{\alpha \in A} U_\alpha$ is connected.*

Proof. Let $Y = \bigcup_{\alpha \in A} U_\alpha$ and suppose that (U,V) is a separation of Y. Let $a \in \bigcap_{\alpha \in A} U_\alpha$. With the love of God, assume that $a \in U$, so each U_α intersects U so $U_\alpha \subset U$ for all $\alpha \in A \implies V$ is empty, a contradiction. ■

Theorem 168 *Let A be a connected subspace of X and let $B \subset X$ such that $A \subset B \subset \overline{A}$. Then, B is connected. In particular, the closure of a connected set is connected.*

Proof. Suppose that (U, V) is a separation of B. Then, $U \subset B$. WLOG, assume $A \subset U$. Then, $\overline{A} \subset \overline{U}$ so that $A \subset B \subset \overline{A} \subset \overline{U}$. That is, $U \subset B \implies B \subset \overline{U} \subset \overline{B} \implies \overline{B} \subset \overline{U}$. Thus, $\overline{B} = \overline{U}$. It must be that $B = U$ and that $V = \emptyset$, a contradiction. ■

We will now prove that $\mathbb{R}$ is connected by showing a stronger statement: intervals in $\mathbb{R}$ are connected. Recall that $\mathbb{R}$ has the least upper bound property. That is, each subset $A \subset \mathbb{R}$ which is bounded above has a least upper bound. A is said to be bounded above if $\exists x \in \mathbb{R}$ such that, $\forall y \in A$, $x \geq y$. x is a least upper bound for A if, given any upper bound z of A, $x \leq z$. Denote the least upper bound of A by $\sup A$.

What are all the connected subsets of $\mathbb{R}$? All intervals.

Lemma 169 *If $a, b \in \mathbb{R}$ with $a < b$, then $[a, b]$ is connected.*

Proof. Assume that $[a, b]$ is not connected. Let (U, V) be a separation of $[a, b]$. In particular, $U \subset [a, b]$ so U is bounded above. Let $c = \sup U$. We now claim that $c \in \overline{U} \subset [a, b]$ and $c \notin U \cup V$. Both of these together lead to a contradiction since $U \cup V = [a, b]$. To show that $c \in \overline{U}$, assume the opposite so that $\exists \epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \cap U = \emptyset$. If $y \in U$, then either $y \geq c + \epsilon$ or $y \leq c - \epsilon$. The first case is impossible since it implies that $y > c$ and for any $y \in U$ but then c is not an upper bound of U. If $y \leq c - \epsilon$ for all $y \in U$, then $c - \epsilon$ is also an upper bound for U but this means that $c \neq \sup U$, another contradiction. Hence $c \in \overline{U}$. We know that $U \subset [a, b]$ and, by definition, $\overline{U} \subset [a, b]$.

For the second claim that $c \notin U \cup V$, recall that $U \cap V = \emptyset \implies \overline{U} \cap V = \emptyset$ by **Theorem 164**. Thus, $c \notin V$. Assume that $c \in U$. Then, U is open $\implies \exists \delta$ such that $(c - \delta, c + \delta) \subset U$, but this is a contradiction since $c < c + \frac{\delta}{2} \in U$ which means that $c \neq \sup U$. ■

Theorem 170 *Intervals in $\mathbb{R}$ are connected*

Proof. Let $I \subset R$ be an interval and suppose that (A, B) is a separation of I. Let $a \in A$ and $b \in B$. Then, $[a, b] \subset I$ but since $[a, b]$ is connected, we know that either $[a, b] \subset A$ or $[a, b] \subset B$ but then $A \cap B \neq \emptyset$, a contradiction. ■

Theorem 171 (Intermediate Value Theorem) *Let $X \subset \mathbb{R}$ be a connected subspace and let $f : X \longrightarrow \mathbb{R}$ be continuous such that $\exists a, b \in X$ such that $f(a) < r$ and $f(b) > r$. Then, $\exists c$ such that $a < c < b$ and $f(c) = r$.*

Proof. We know that $f(X)$ is connected because f is continuous. We also know that $f(a), f(b) \in f(X)$. Assume, for contradiction, that $\forall c \in X$, $a \geq c$ or $c \geq b$ or $f(c) \neq r$. Then, $r \notin f(X)$. Let $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, \infty)$. Then, A and B are open and disjoint. A is non-empty since $f(a) \in A$. Similarly, B is non-empty since $f(b) \in B$. That is, $f(X)$ is separated, a contradiction. Hence $r \in f(X)$ and we have a $c \in X$ such that $f(c) = r$ and $a < c < b$. ■

6.1 Product of Connected Sets

Is $\mathbb{R}^{\mathbb{N}}$ connected? Depends on the topology. In the box topology, $\mathbb{R}^{\mathbb{N}}$ is not connected: Let $A = \{\text{bounded sequences}\}$ and $B = \{\text{unbounded sequences}\}$. Then, A and B are non-empty, disjoint. We only need to show that A and B are open. Let $x = (x_1, x_2, ...) \in \mathbb{R}^{\mathbb{N}}$ and let $U = (x_1 - 1, x_1 + 1) \times (x_2 - 1, x_2 + 1) \times ....$ In the box topology, this is open (but not in product topology!). Then, $x \in U$ and if x is bounded (resp. unbounded), then U consists of bounded (resp. unbounded) sequences. Thus x is an interior point of A (resp. B). Since x was arbitrary, therefore A (resp. B) is open.

Lemma 172 *If X and Y are topological spaces and $A \subset X$ and $B \subset Y$ are connected, then $A \times B$ is connected in either box or product topology.*

Proof. $A \times B = \bigcup\limits_{x \in A} T_{(x,b)}$ where $T_{(a,b)} = (\{a\} \times B) \cup (A \times \{b\})$. We claim that $T_{(x,b)}$ is connected for any $x \in A$. Clearly, $\{x\} \times B$ is connected, as is $A \times \{b\}$. Also, $(\{x\} \times B) \cap (A \times \{b\}) = \{(x, b)\}$, always, so that there is no separation of $T_{(x,b)}$ so that $T_{(x,b)}$ is connected. For all x, $T_{(x,b)}$ contains (a, b) so that $A \times B = \bigcup\limits_{x \in A} T_{(x,b)}$. ■

Corollary 173 *Finite product of connected sets is connected.*

Theorem 174 $\mathbb{R}^{\mathbb{N}}$ *is connected in product topology*

Proof. $\forall n \in N$, let $\widetilde{\mathbb{R}^n} = \{(x_1, x_2, ...) : x_i = 0 \text{ for } i > n\}$. $\widetilde{\mathbb{R}^n}$ is homeomorphic to $\mathbb{R}^n$ via $(x_1, x_2, ...) \longmapsto (x_1, x_2, ..., x_n, 0, 0, ...)$. $\widetilde{\mathbb{R}^n}$ is connected for each n. Let $X = \bigcup\limits_{n \in \mathbb{N}} \widetilde{\mathbb{R}^n}$. X is connected since each $\widetilde{\mathbb{R}^n}$ is connected and they all contain $(0, 0, ...)$. Claim: $\overline{X} = \mathbb{R}^{\mathbb{N}}$. It is obvious that $\overline{X} \subset \mathbb{R}^{\mathbb{N}}$. Let $x \in \mathbb{R}^{\mathbb{N}}$ and let U be a neighborhood of x. In the product topology, $U = \prod\limits_{i \in \mathbb{N}} U_i$ where $U_i = \mathbb{R}$ for all but finitely many $i \in \mathbb{N}$. Let $k = \max\{i : U_i \neq \mathbb{R}\}$. Let $p = (x_1, x_2, ..., x_k, 0, 0, ...)$. Then, $p \in \widetilde{\mathbb{R}^k}$ and $p \in U$ since $U_i = \mathbb{R}$ for $i > k$. So, any neighborhood U with $x \in U$ intersects some $\widetilde{\mathbb{R}^k}$ and hence intersects X, so $x \in \overline{X}$ ■

Theorem 175 *Arbitrary Cartesian Product of connected sets is connected in the product topology/*

Proof. Proof follows a pattern similar to connectedness of $\mathbb{R}^n$. Let $\{X_\alpha, \tau_\alpha\}_{\alpha \in A}$ be a collection of topological spaces and $X = \prod\limits_{\alpha \in A} X_\alpha$. Pick $a \in X$. For each finite $F \subset A$, define $X_F = \{x \in X : \pi_\beta(x) = \pi_\beta(a) \forall \beta \notin F\}$. It can be shown that X_F is homeomorphic to $\prod\limits_{\alpha \in F} X_\alpha$ and that X_F is connected. Now let $Y = \bigcup\limits_{\substack{F \subset A \\ |F| < \infty}} X_F$. Then, each X_F contains a so Y is connected. Finally, $\overline{Y} = X$. ■

6.2 Path Connected

Definition 176 *Let X be a topological space and let $a, b \in X$. Then, a* ***path*** *in X from a to b is a continuous function $f : [c, d] \longrightarrow X$ such that $f(c) = a$, $f(d) = b$. X is said to be* ***path connected*** *if there is a path between any two points.*

Example 177 *$\mathbb{R}^n$ is path connected. Let $x, y \in \mathbb{R}^n$. Then, we can define a path $f : [0,1] \longrightarrow \mathbb{R}^n$ as follows: $t \longmapsto (1-t)x + ty$.*

Example 178 *$N_\epsilon(0) \subset \mathbb{R}^n$ is also path connected. Let $x, y \in N_\epsilon(0)$. Then, we can define a path $f : [0,1] \longrightarrow \mathbb{R}^n$ as follows: $t \longmapsto (1-t)x + ty$ because $|(1-t)x + ty| \leq |(1-t)x| + |ty| < (1-t)\epsilon + t\epsilon = \epsilon$. $N_\epsilon(0)$ is homeomorphic to $N_\epsilon(z)$ for any $z \in \mathbb{R}^n$ so that $N_\epsilon(z)$ is also path connected. In fact, this proof works even for l^p spaces*

Example 179 *The sphere $S^n \subset \mathbb{R}^{n+1}$ centred at the origin is also connected. Let $x, y \in S^n$ and P be the plane which contains x, y. Let $z \in P \cap S^n$ such that $x \perp z$. Write points in $P \cap S^n$ as $\cos tx + \sin tz$. Then, $\exists T \in [0, 2\pi]$ such that $y = \cos Tx + \sin Tz$ so that we can have a function $f : [0,1] \longrightarrow S^n$ such that $t \longmapsto \cos tx + \sin tz$.*

Example 180 *$\mathbb{R}^n \backslash \{x\}$ is path connected for $n > 1$. We can take $x = 0$ since $\mathbb{R}^n \backslash \{x\}$ is homeomorphic to $\mathbb{R}^n \backslash \{0\}$ under the mapping $y \longmapsto y - x$. Then, for any x, y, we can have a path between $\widetilde{x} = \frac{x}{|x|}$ and $\widetilde{y} = \frac{y}{|y|}$ as above.*

Theorem 181 *If X is path connected, then X is connected.*

Proof. Assume that X is not connected, so that we have a separation (U, V). Let $a \in U$ and $b \in V$. Since X is path connected, $\exists f : [c, d] \longrightarrow X$ such that $f(c) = a$, $f(d) = b$. However, $f([c, d])$ is connected so that either $f([c, d]) \subset U$ or $f([c, d]) \subset V$, but then $b \in U$ or $a \in V$. In either case, $U \cap V \neq \emptyset$, a contradiction. ■

The converse doesn't follow:

Example 182 *Let $I = [0,1]$, $X = I \times I$ with order topology. X is connected in the usual topology. X is also connected in the product topology. Let (U, V) be a seperation of X.* ***Then..?***

However, X is not path connected. Suppose that X is path connected. Let $p = (0,0)$ and $q = (1,1)$. Since X is assumed to be path connected, there is a path $f : [a, b] \longrightarrow X$ connecting p and q. Note that $f(a) = (0,0)$ and $f(b) = (1,1)$ by definition of path and that $(0,0) < (1,1)$. Thus, $f(a) < f(b)$. Now let $d \in X$. Then, $p \leq d \leq q$. Hence $\exists c \in [a, b]$ such that $f(c) = d$ by IVT. That is, f is surjective. Now let $x \in I$ and $V_x = \{(x, y) : y \in I\}$ is an interval in X and is open. Thus, $U_x = f^{-1}(V_x)$ is open in $[a, b]$. If $y \neq x$, then $V_x \cap V_y$ is disjoint and so is $U_x \cap U_y$. Now for each $x \in I$, pick an element $r_x \in U_x \cap \mathbb{Q}$. Thus, we can have a well-defined function $F : I \longrightarrow \mathbb{Q}$ defined by $f(x) = r_x$. F is injective: $f(x) = f(y) \implies r_x = r_y$. Since $r_x \in U_x \cap \mathbb{Q}$ and

$r_y \in U_y \cap \mathbb{Q}$ and $U_x \cap U_y = \emptyset$ for all distinct x, y, then $r_x = r_y \implies x = y$. Since F is an injection, we have a copy of an uncountable set in a countable set, a contradiction.

Existence of paths is transitive: if $f : [a, b] \longrightarrow X$ be a path between x and y and $g : [c, d] \longrightarrow X$ be another path between y and z. That is, $f(a) = x$, $f(b) = y = g(c)$ and $g(d) = z$. Then, replace g with $\widetilde{g} : [b, d' = d + b - c] \longrightarrow X$. Then, $\widetilde{g}(x) = g(x - b + c)$. We can now concatenate f and g as follows: $\widetilde{g} \circ f = f(x)$ if $x \in [a, b]$ and $= \widetilde{g}(x)$ if $x \in [b, d]$.

In **Theorem 168**, we've seen that the closure of a connected set is connected. However, the same doesn't hold for path connectedness:

Example 183 *Topologist's sine curve: $S = \left\{\left(x, \sin \frac{1}{x}\right) : x \in (0, 1]\right\}$, the graph of $h(x) = \sin\left(\frac{1}{x}\right)$. S is path connected since between any two points x, y in S, we can define a continuous function by simply taking the restriction of h to those two points. Since S is path connected, it is connected. Take $\overline{S}$, which is connected because S is connected. Note that $\overline{S} = S \cup V$ for $V = \{(0, y) : y \in [-1, 1]\}$. However, $\overline{S}$ is not path connected because we can't find a point from the vertical axis V to the curve. Assume that we have a path $f : [a, b] \longrightarrow \overline{S}$ with $f(a) = (0, 0)$ and $f(b) = q \in S$. V is a closed set in $\overline{S}$ so that $f^{-1}(V) \subset [a, b]$ is also closed. Then, $\sup f^{-1}(V) = c \in f^{-1}(V)$. Now form $g : [c, b] \longrightarrow \overline{S}$ a path where every point other than c maps into S. This is like taking just the last point where the original function f meets V. WLOG, assume that $[c, b] = [0, 1]$. Now let $f(t) = (x(t), y(t))$ with $x(0) = 0$, $x(t) > 0$ for $t > 0$ and $y(t) = \sin \frac{1}{x(t)}$ if $t > 0$. The idea now is to find a sequence $t_n \longrightarrow 0$ but that $f(t_n)$ does not converge to $f(0)$. For each n, choose u so that $0 < u < x\left(\frac{1}{n}\right)$ and $\sin \frac{1}{u} = (-1)^n$. We know that $x(t)$ is continuous so that by IVT, we can pick $t_n \in \left(0, \frac{1}{n}\right)$ such that $x(t_n) = u$. Notice that $t_n \longrightarrow 0$ but that $f(t_n) = (x(t_n), y(t_n)) = (x(t_n), (-1)^n)$ does not converge.*

6.3 Connected Components

Let X be a topological space. There is a relation on X given by $a \sim b$ if $\exists$ a connected $A \subset X$ such that $a, b \in A$. $\sim$ is an equivalence relation.

Proof. $\sim$ is symmetric: Let $x \in X$. Consider the subset $\{x\} \subset X$. Equip this with the subspace topology. Then, the subspace topology for this singleton is indiscrete and, therefore, the only non-empty open and closed subset of $\{x\}$ is $\{x\}$ itself. Hence $\{x\}$ is a connected subspace of X. Thus, $x \sim x$.

$\sim$ is reflexive: For $x, y \in X$, let $x \sim y$. Then, there exists a connected subspace $A \subset X$ such that $x, y \in A$. That is, there exists a connected subspace $A \subset X$ such that $y, x \in A \implies y \sim x$

$\sim$ is transitive: For $x, y, z \in X$ let $x \sim y$ and $y \sim z$. Then, there exist connected subspaces $A, B \subset X$ such that $x, y \in A$ and $y, z \in B$. Since $y \in A \cap B$, then $A \cap B \neq \emptyset$ so that $A \cup B$ is a connected subspace of X, containing x, y, z. Thus $x \sim z$ ■

Thus, if X is connected, there is a single equivalence class.

Definition 184 *The equivalence classes of $\sim$ are called **connected components** of X.*

Theorem 185 *The connected components of X are disjoint, connected subsets of X whose union is all of X. Furthermore, each non-empty connected subset of X is contained in a unique connected component.*

Proof. First part is obvious, considering that the equivalence class forms a partition. To show that the connected components are connected, let A be a connected component. Let (U, V) be a separation of A. Let $a \in U$ and $b \in V$. Since $a, b \in A$, there exists connected $C \subset X$ such that $a, b \in C$. Then, $C \subset U$ or $C \subset V$ but then $b \in U$ or $a \in V$, a contradiction to the fact that $U \cap V = \emptyset$.

If B is a connected, non-empty subset and $b \in B$, then b intersects the connected component containing b but since B is connected, then B must be contained in this component. ■

Theorem 186 *Let (X, τ) be a topological space. Then,*

1. *The connected components are closed.*
2. *If there are finitely many components, then the union of the components is closed, as well.*
3. *If there are finitely many components, then the components are also open.*

Proof. (1) Let U be a component. Then, U is connected and so is $\overline{U}$. The latter lives in some component and that component certainly has to be U. Thus, $U = \overline{U}$

(2) Let Q be the union of the connected components other than U, so Q is a union of finitely many closed sets and hence Q is closed.

(3) **Proof?** ■

Example 187 $X = [-1, 0) \cup (0, 1]$. *The connected components are* $U_1 = [-1, 0)$ *and* $U_2 = (0, 1]$*, which are both open and closed in the subspace topology.*

Example 188 *Let $\mathbb{Q}$ have the subspace topology of $\mathbb{R}$. The connected components are singletons, which are closed but not open: let $U \subset \mathbb{R}$ be an open subset. Then, U contains infinitely many rationals.*

6.4 Path Components

Definition 189 *Let (X, τ) be a topological space. Then, there is another equivalence relation $\equiv$ on X where $x \equiv y \iff$ if there is a path in X between x and y. The equivalence classes of $\equiv$ are called **path components**.*

Theorem 190 *Let (X, τ) be a topological space. The path connected components of X are disjoint, connected subsets of X whose union is all of X. Furthermore, each non-empty path connected subset of X is contained in a unique connected component.*

Proof. Since path connectedness implies connectedness, then every path connected component is a connected component. Hence the theorem holds. ∎

Example 191 *In **Example 183**, $\overline{S}$ is connected but not path connected and S and V are the path components of $\overline{S}$. Also, S is open but not closed. V is closed but not open. Thus, path components are neither guaranteed to be open nor closed, even in the finite case.*

6.5 Local and Path Connectivity

Definition 192 *Let (X, τ) be a topological space and let $y \in X$. X is said to be locally connected at y if for every neighborhood U of y, $\exists$ a connected open set $y \in V \subset U$. X is **locally connected** if it is locally connected at each point.*

Definition 193 *Let (X, τ) be a topological space and let $y \in X$. X is said to be path connected at y if for every neighborhood U containing y, $\exists$ a path connected open set $y \in V \subset U$. X is **path connected** if it is path connected at each point.*

Example 194 *The Sine Curve, at the origin **(isn't the sine curve undefined at the orgin?)**, is connected but not locally connected.*

Example 195 $X = [-1, 0) \cup (0, 1]$. *Then, X is not connected but is locally connected.*

The above examples show that connected and locally connected do not have any logical connection.

Example 196 *Let S be the topologist sine curve union and $I \subset \mathbb{R}^2$ be any open set far away from S. Then, $\overline{S} \cup I$ is neither connected nor locally connected*

Theorem 197 *X is locally connected if and only if for each open set U in X, every component of U is open. In particular, every component is open.*

Proof. Suppose that X is locally connected and let $U \subset X$ be open. Let $y \in U$ and C be a component of U that contains y. We need to show that C is open. By local connectedness, we know that $\exists$ a connected, open V and so that $y \in V \subset U$ but since V is connected, we must have $y \in V \subset C$ so that $y \in Int\,(C)$. Since y was arbitrary, C is open.

Conversely, let $y \in X$ and let components of open sets be open. Let $U \subset X$ be an open set with $y \in U$. Then, y is contained in some component $C \subset U$ but C is open and connected so X is locally connected at y. Since y was arbitrary, X is locally connected. ∎

Remark 198 *The same theorem is true for path components.*

Theorem 199 *Let (X, τ) be a topological space. Then, each path component lies in a unique component of X. Furthermore, if X is locally path connected. Then, the path components and the components are the same.*

Proof. Let $P \subset X$ be a path component. Then, P is path connected and hence connected. So, there is a component $C \subset X$ such that $P \subset C$. Since X is locally path connected, P is open. Let Q be the union of all path components intersecting C other than P. Then, Q is a union of path components and is hence open, showing that Q is open and $C \backslash Q = P$ is closed. Thus, P is a non-empty clopen set and so $P = C$. ■

Corollary 200 *If X is locally path connected, then X is connected $\Longleftrightarrow$ X is path connected.*

In summary, path components and components are different in sine curve.

7 Compact Spaces

7.1 Compactness in Topological Spaces

A topological space is compact if it is well-approximated by finite sets. In formal speak, a let X be a topological space and let C be a collection of open subsets. If $X = \bigcup\limits_{A \in C} A$, then C is called an **open cover** of X. X is said to be **compact** if each open cover contains a finite subcollection that is also a cover. Colloquially, every cover has a finite subcover.

Example 201 *Any finite set with any topology is compact. Let $X = \{x_1, x_2, ...x_n\}$ and let C be an open cover. Then, for each $i \in \{1, ..., n\}$, $\exists x_i \in U_i \in C$ so $\{U_1, ..., U_n\}$ is a finite subcover.*

Example 202 *If X is any set with indiscrete topology, then $C = \{X\}$ is the only open cover, which is already finite. Any discrete topology is finite if and only if the underlying set is finite. We've already shown that if a set is finite with the discrete topology, then it is compact. To show the converse, let $|X| = \infty$ and $C = \{\{x\} : x \in X\}$. This is a cover and has no proper subcover.*

Example 203 *Let $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\} \subset \mathbb{R}$ be a sequence. Let C be a cover of A. Then $\exists U \in C$ such that $0 \in U$ but we know that $\frac{1}{n} \longrightarrow 0$ so that $\exists N \in \mathbb{N}$ so that if $n \geq N$, then $\frac{1}{n} \in U$. Infinitely many points are now placed in a single set. For the remaining finite points $1 \leq m < n$, $\exists U_m \in C$ such that $\frac{1}{m} \in U_m$ because C is a cover. Hence the subcollection $\{U, U_1, U_2, ..., U_{N-1}, U\}$ is a subcover.*

This argument works even for non-Hausdorff spaces!

$\mathbb{R}$ is not compact. Let $C = \{(n, n+2) : n \in \mathbb{Z}\}$. Removing a single interval will miss a real number, causing the subcollection to not be a cover.

$(0, 1)$ is not compact because it is homeomorphic to $\mathbb{R}$ and compactness is a topological property.

$(0, 1]$ is also not compact. Let $C = \left\{\left(\frac{1}{n}, 1\right] : n \in \mathbb{N}\right\}$ be an open cover. Assume it has a subcover, C'. Then, there is a largest m so that $\left(\frac{1}{m}, 1\right]$. It is easy to see that $\frac{1}{2m}$ is not in any $U \in C'$.

Theorem 204 *If $a < b$ are real numbers, then $[a, b]$ is compact.*

The proof will proceed as follows: we will first let C be a cover of $[a, b]$ and then create a set

$$D = \{y \in (a, b] : [a, y] \text{ is covered by finitely many elements of } C\}$$

Then, for $d = \sup D$, we will show that $d = b$.
Proof. Claim: $D \neq \emptyset$. Choose $U \in C$ so that $a \in U$. Since U is open, $\exists z \in (a, b)$ such that $[a, z] \subset U$. Then, $[a, z]$ is covered by a single element of C so that $z \in D$. Now we can define the supremum. Let $d = \sup D$. We first show that $d \in D$. Choose $V \in C$ such that $d \in V$. Since V is open, then V contains an interval $(c, d]$ where $c \in [a, d)$. Assume that $c \notin D$. Then, there is a $z \in (c, d) \cap D$ **(what?)** otherwise $d \neq \sup D$. Since $z \in D$, $[a, z]$ can be covered by finitely many elements $\{C_1, ..., C_n\}$ of C and so $[a, d]$ is covered by $\{C_1, ..., C_n, V\}$

Finally, we need to show that $d = b$. $d > b$ is false, otherwise $d \neq \sup D$. Assume $d < b$. Arguing as above, we can find y so that $d < y \leq b$ so that $[d, y]$ is contained in a single element $U \in C$. Since $d \in D$, then $[a, d]$ is covered by finitely may elements $\{C_1', ..., C_n'\}$ of C and so $[a, y]$ is covered by $\{C_1', ..., C_n', U\}$ but this contradicts that $d = \sup D$. Hence $d = b$ and $[a, b]$ is covered by finitely many elements of C and hence compact. ■

Theorem 205 *If X is compact and $A \subset X$ is closed. Then A is also compact.*

Proof. Let C be a cover of A. Since A is closed, $X \backslash A$ is open and so $C' = C \cup \{X \backslash A\}$ is a cover of X. Since X is compact, we have a finite subcover B of X. We now show that $B \cup \{X \backslash A\}$ is a finite cover of A. If $a \in A$, then $a \in U \in C'$. Then, $U \in C$ or $U = X \backslash A$. If $U = X \backslash A$, $a \in X \backslash A$, a contradiction. Thus, $U \in C$ so that $\exists U' \in B$ such that $a \in U'$. ■

Theorem 206 *If X is a Hausdorff topological space and $A \subset X$ is compact, then A is closed.*

Proof. Let $y \in X \backslash A$. For each $a \in A$, there exists disjoint nbds U_a and V_a with $y \in U_a$ and $a \in V_a$. Let $C = \{V_a : a \in A\}$. This is a cover for A. Since A is compact, C has a finite subcover. $C' = \{V_{a_1}, V_{a_2}, ..., V_{a_n}\}$. Then, $V = \bigcup_{\Lambda \in C'} \Lambda$ is open, contains A and is disjoint from $U = U_{a_1} \cap U_{a_2} \cap ... \cap U_{a_n}$. Each set here is open and contains y so that U is open and contains y and $U \subset X \backslash A$. Hence y is an interior point of $X \backslash A$. y was arbitrary. Therefore, $X \backslash A$ is open or that A is closed. ■

Corollary 207 *If Y is a compact subspace of a Hausdorff space X and $a \in X \backslash Y$. Then, there are disjoint open sets U and V so that $a \in U$ and $Y \subset V$.*

Theorem 208 *Let (X, τ_X) be a compact topological space and $f : X \longrightarrow Y$ be continuous. Then, $f(X)$ is compact*

Proof. Let C be a cover of $A = f(X)$. Let $C' = \left\{ f^{-1}(U) : U \in C \right\}$. It is easy to see that C' is an open cover of X. Since X is compact, C' has a finite subcover $C'' = \{V_1, ..., V_n\}$. For each $V_i \in C''$, $\exists U_i \in C$ such that $V_1 = f^{-1}(U_i)$. We now show that $\{U_1, ..., U_n\}$ is a finite subcover. It is clearly finite and a collection of open sets. To show that it is a cover, let $d \in A$. Then, $\exists e \in X$ such that $f(e) = d$. That is, $\exists i \in \{1, ..., n\}$ so that $e \in V_i$. Thus, $d \in U_i$. ■

Theorem 209 *Let X be compact and Y be Hausdorff. If $f : X \longrightarrow Y$ is a continuous bijection, then f is a homeomorphism.*

Proof. Let $g = f^{-1}$. We need to show that g is continuous by showing that preimages of closed sets are closed. Let $A \subset X$ be a closed set. Then, A is compact. Since f is a bijection, $g^{-1}(A) = f(A)$. Since f is continuous, $f(A)$ is also compact. However, Y is Hausdorff. So, $f(A) = g^{-1}(A)$ is closed. ■

Lemma 210 (Tube Lemma) *If X and Y are compact and $N \subset X \times Y$ is an open set that contains $\{a\} \times Y$ for some $a \in X$. Then, there exists neighborhood W of a such that $W \times Y \subset N$.*

Proof. $Y_a = \{a\} \times Y$ is homeomorphic to Y and hence compact. Let $p \in Y_a$. Then, $p = (a, y)$ for some $y \in Y$. Pick $U_p \subset X$ and $V_p \subset Y$ so that $p \in U_p \times V_p \subset N$. Let $C = \{U_p \times V_p : p \in Y_a\}$. This is an open cover of Y_a so has a finite subcover $\{U_{p_1} \times V_{p_1}, ..., U_{p_n} \times V_{p_n}\}$. WLOG, assume that $a \in U_{p_i}$ for every $i \in \{1, ...n\}$. Let $W = \bigcap_{i=1}^{n} U_{p_i}$. Then, $W \times Y \subset N$ because $W \times Y = \bigcup_{i=1}^{n} W \times V_{p_i}$ and $W \times V_{p_i} \subset U_{p_i} \times V_{p_i} \subset N$ ■

Theorem 211 (Weak Tychonoff's Theorem) *Let X and Y be compact. Then, $X \times Y$ is compact.*

Proof. Let C be a cover of $X \times Y$ and let $a \in X$. Then, Y_a is compact and so Y_a is covered by finitely many elements $\{A_1, ..., A_n\}$ of C. Let $N = \bigcup_{i=1}^{n} A_i$ be an open set that contains Y_a. By Tube Lemma, $\exists W_a \subset X$ such that $W_a \times Y \subset N$ so that W_a is covered by finitely many elements of C. The collection $\{W_a : a \in X\}$ is an open cover of X and so has a finite subcover, say $\{W_1, ..., W_n\}$. We can write $X \times Y = \bigcup_{i=1}^{n} W_i \times Y$. Each $W_i \times Y$ is covered by finitely many elements of C and so C has a finite subcover. ■

Corollary 212 *If X_i is compact for $1 \leq i \leq n$, then $X_1 \times ... \times X_n$ is compact.*

Definition 213 *Let (X, d) be a metric space. A set $A \subset X$ is **bounded** if $\exists y \in X$ and $r \in \mathbb{R}$ such that $A \subset N_r(y)$*

Boundedness is a metric property! And crucially depends on metric. To say that for $A \subset R^n$, A is bounded in some l^p metric is equivalent to saying that A is bounded in any l^p metric. This can be proved using the fact that for any $p \geq 1$, there exists a real c_p such that $\frac{1}{c_p} d_p(x,y) \leq d_2(x,y) \leq c_p d_p(x,y)$

Theorem 214 (Heine-Borel Theorem) *$A \subset \mathbb{R}^n$ is compact if and only if A is closed and bounded in some l^p metric for $p \geq 1$.*

Proof. ($\Longrightarrow$) Suppose that $A \subset \mathbb{R}^n$ is compact. $\mathbb{R}^n$ is Hausdorff so A is closed. Consider the collection $\{N_n(0) : n \in \mathbb{N}\}$. Then, this is a cover of $\mathbb{R}^n$ and hence covers A. There exists $m \in \mathbb{N}$ such that $\{N_1(0), ..., N_m(0)\}$ is a cover of A and that $A \subset N_m(0)$. Hence A is bounded.

($\Longleftarrow$) If A is bounded, $\exists x \in \mathbb{R}^n$, $r \in \mathbb{R}$ such that $A \subset N_r(x)$. Observe that if $y \in A$, then $d(0,y) \leq d(0,x) + d(x,y) \leq d(0,x) + r = r'$, say. Hence $A \subset N_{r'}(0)$. Note that $N_{r'}(0) \subset [-r', r']^n = Q$, say. By weak Tychonoff theorem, Q is compact and $A \subset Q$ is closed and so A is compact. ■

Theorem 215 (Extreme Value Theorem) *Let $f : X \longrightarrow \mathbb{R}$ be continuous. If X is compact, then $\exists c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for any $x \in X$.*

Proof. We know that $f(X) \subset \mathbb{R}$ is compact and hence closed and bounded. Let $M = \sup f(X)$ and $m = \inf f(X)$. Since $f(X)$ is closed, $M, m \in f(X)$ hence there exists c, d such that $f(c) = m \leq f(x) \leq f(d) = M$ for any $x \in X$ ■

There is another definition of compactness:

Definition 216 *Let C be a collection of subsets of a set X. Then, C has the* ***finite intersection property*** *(FIP) if for each finite subcollection $\{C_1, ..., C_n\} \subset C$, $\bigcap_{i=1}^{n} C_i \neq \emptyset$*

This definition does not require X to be a topological space

Theorem 217 *Let X be a compact space. Then, X is compact iff every collection of closed subsets of X with FIP has the property that $\bigcap_{D \in C} D \neq \emptyset$*

Proof. A is collection of open sets of X iff $C = \{X \backslash U : U \in A\}$ is a collection of closed subsets of X. A is a cover iff $\bigcap_{U \in C} U = \varnothing$. Next, a finite subcollection $\{A_1, ..., A_n\} \subset A$ is a subcover iff $\bigcap_{i=1}^{n} X \backslash A_i \neq \emptyset$. Finally, a collection of closed sets has the FIP iff the corresponding open sets contains no finite cover. ■

Theorem 218 (Tychonoff's Theorem) *If $\{X_\alpha\}_{\alpha \in A}$ be a collection of compact topological spaces, then $\prod_{\alpha \in A} X_\alpha$ is compact in product topology.*

The proof of this fact uses Zorn's lemma (if every chain in a poset P has an upper bound, then P has a maximal element)

Consider the following doomed proof: We need to show that $X = \prod_{\alpha \in A} X_\alpha$ is compact. Let C be a collection of closed sets with FIP. For each $\alpha \in A$, define $C_\alpha = \left\{ \overline{\pi_\alpha (D)} : D \in C \right\}$. This is a collection of closed sets in X_α. It is easy to check that D_α has the FIP. Since X_α is compact for each α, there is $x \in \bigcap_{D \in C_\alpha} D$. Let $x = (x_\alpha)_{\alpha \in A}$ with $\pi_\alpha (x) = x_\alpha$. However, $x \notin \bigcap_{D \in C} D$. Say $X = [0,1] \times [0,1]$. Let $p = (x_1, y_1)$ and $q = (x_2, y_2)$. C be the collection of closed ellipses ith foci p and q.

To prove Tychonoff's theorem, we will use the following lemma:

Lemma 219 *Let X be a set and let C be a collection of non-empty sets with FIP. Then, there is a collection of non-empty set D such that*

1. *$C \subset D$*
2. *D has the FIP*
3. *$D \subsetneq E \implies$ then E does not FIP*

Proof. Let C^* be a collection of collection of non-empty subsets of X that contain C and have the FIP. Then, C^* is partially ordered by inclusion. Suppose $B^* \subset C^*$ is a chain and let $A = \bigcup_{b \in B^*} b$. Then, any $b \in B^*$ is a subset of A. Clearly, $C \subset A$ since $C \subset b$ for any $b \in B^*$. We claim that A has the finite intersection property. Let $\{\alpha_1, ..., \alpha_n\}$ be a finite subcollection of A. For each $1 \leq i \leq n$, there is a collection $b_i \in B^*$ such that $\alpha_i \in b_i$. Then, $\{b_1, ..., b_n\} \subset B^*$. WLOG, we can assume that $\{b_1, ..., b_n\}$ is ordered in a way such that $b_i \leq b_{i+1}$ and so b_n is a maximal element of $\{b_1, ..., b_n\}$ so that $\alpha_i \in b_n$ for $1 \leq i \leq n$. So, $\{\alpha_1, ..., \alpha_n\}$ is a finite subcollection of $b_n \in B^* \subset C^*$ so $\bigcap_{i=1}^{n} \alpha_i \neq \emptyset$. So, A has the FIP and so $A \in C^*$ so all chains have an upper bound. By Zorn's lemma, C^* has a maximal element in D. We can then use the following: ■

Lemma 220 *Let X be a set and let C be a collection of non-empty subsets with FIP and let D be the maximal FIP collection containing C. Then,*

1. *Any finite intersection of elements of D is in D*
2. *If $A \subset X$ and A intersects each element of D, then $A \in D$*

Proof. Let $\{D_1, ..., D_n\}$ be a finite subcollection of D and let $B = \bigcap_{i=1}^{n} D_i$. Since D has FIP, $B \neq \emptyset$. Let $E = D \cup \{B\}$. Let $\{E_1, ..., E_k\}$ be a finite subcollection of E such that $\bigcap_{i=1}^{k} E_i = \emptyset$. Then, $B \in \{E_1, ..., E_k\}$ for otherwise, if $B \notin \{E_1, ..., E_k\}$, then $\bigcap_{i=1}^{k} D_i = \emptyset$. WLOG, assume that $B = E_k$ and $E_j \neq B$ for $j < k$. Then, $\bigcap_{i=1}^{k} E_i = \bigcap_{i=1}^{k-1} E_i \cap B = \bigcap_{i=1}^{k-1} E_i \cap \bigcap_{i=1}^{n} D_i$ but since $\{E_1, ..., E_{k-1}, D_1, ..., D_n\}$ is a finite subcollection of D and so, $\bigcap_{i=1}^{k} E_i \neq \emptyset$, a contradiction.

For 2, let $F = D \cup \{A\}$. Let $\{F_1, ..., F_n\}$ be a finite subcollection of F such that $\bigcap_{i=1}^{n} F_i$. Again, $A \in \{F_1, ..., F_n\}$ (see above) so that WLOG $A = F_n$ and $F_i \neq A$ for $i < n$. Then, $\bigcap_{i=1}^{n} F_i = \bigcap_{i=1}^{n-1} F_i \cap A \neq \emptyset$, a contradiction. ■

And now, for the proof of Tychonoff's theorem

Proof. Let $X = \prod_{\alpha \in A} X_\alpha$ and let C be a collection of non-empty subsets with FIP. We will prove that $\bigcap_{\Omega \in C} \overline{\Omega} \neq \emptyset$. Let D be the maximal FIP collection containing C. We will show that $\emptyset \neq \bigcap_{\Omega \in D} \overline{\Omega} \subset \bigcap_{\Omega \in C} \overline{\Omega}$. Let $\alpha \in A$ and $\pi_\alpha : X \longrightarrow X_\alpha$ and consider the collection $D_\alpha = \{\pi_\alpha (\delta) : \delta \in D\}$. Then, D_α has FIP since D does. Since X_α is compact, we can choose $X_\alpha \in \bigcap_{\delta \in D_\alpha} \overline{\delta}$. Let $x = (x_\alpha)_{\alpha \in A}$. We claim that $x \in \bigcap_{\delta \in D} \overline{\delta}$. Let $B' = \left\{\pi_\alpha^{-1} (U_\alpha) : U_\alpha \subset X_\alpha \text{ is open}\right\}$. Then, B' is a subbase for the product space. Let B be the standard basis for the product topology. Let $\delta \in D$ and let $V = \pi_\alpha^{-1} (U_\alpha)$ be an element of B that contains X. Then, U_B is a neighborhood of $x_\beta = \pi_\beta (x)$. By construction, $x_\beta \in \overline{\pi_\beta (\delta)}$ and U_B intersects $\pi_\beta (\delta)$ in an element of form $\pi_\beta (y)$ where $y \in \delta$. Thus, $y \in V \cap \delta$. In other words, every element of B' that contains x intersects every element of δ so $V \in D$. By previou lemma, every element of B that intersects x is contained in D.

Let W be a basis element containing X. Then, $W \in D$ and so $\{W, \delta\} \subset D$ and by FIP, $W \cap \delta \neq \emptyset$ and $x \in \overline{\delta}$ for all δ. ■

7.2 Compactness in Metric Spaces

Lemma 221 (Lebesgue Number Lemma) *Let (X, d) be a compact metric space and let C be an open cover. Then, $\exists \delta > 0$ such that if $U \subset X$ is an open set of diameter $< \delta$, then $\exists V \in C$ such that $U \subset V$.*

Such a δ is called a **Lebesgue Number** of C **(how is this unique to each C?)**

Proof. Since X is compact, we can find a finite subcover $\{C_1, ..., C_n\}$. Let $\delta = diam\,(C_1)$. Let $A_i = \overline{X - C_i}$.

We will look at the average distance to A_i. Define $f : X \longrightarrow \mathbb{R}$ with $f(x) = \frac{1}{n}\sum_{i=1}^{n} d(x, A_i)$. For each i, $\exists a_i \in A$ such that $d(x, A_i) = \inf\{d(x, a) : a \in A_i\} = d(x, a_i)$. This function is continuous: to show this, we need to show that $x \longmapsto d(x, A_i)$ is continuous. Let $\epsilon > 0$. Let $y \in N_\epsilon(x)$. Then,

$$\begin{aligned} d(y, A_i) &\leq d(y, a_i) \\ &\leq d(y, x) + d(x, a_i) \\ &= d(y, x) + d(x, A_i) \\ &< \epsilon + d(x, A_i) \end{aligned}$$

so that $|d(x, A_i) - d(y, A_i)| < \epsilon = \delta'$. Thus, $x \longmapsto d(x, A_i)$ is continuous. Therefore, $f(x)$ is continuous. Also, $f(x) > 0$ for all x. Since X is compact, by extreme value theorem, $\exists\delta > 0$ such that $f(x) \geq \delta > 0$. We show that this is the Lebesgue number of C. Assume that $diam\,(U) < \delta$. Let $x \in U$ so that $U \subset N_\delta(x)$. Also, $f(x) \geq \delta$ so that there exists i, with $1 \leq i \leq n$ such that $\delta < f(x) \leq d(x, A_i)$ and so $U \subset N_\delta(x) \subset C_i$. ∎

Theorem 222 *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \longrightarrow Y$ be a function. Assume that (X, d_X) is compact. Then, f is continuous if and only if f is uniformly continuous.*

Proof. ($\Longrightarrow$) Let f be continuous, $\epsilon > 0$ and $C' = \left\{N_{\frac{\epsilon}{2}}(y) : y \in Y\right\}$. Then, C' is a cover of y so that $C = \left\{f^{-1}(U) : U \in C'\right\}$ is an open cover of X. Let $\delta > 0$ be a Lebesgue number of C. If $d_X(x_1, x_2) < \delta$, then $\{x_1, x_2\}$ has diameter$< \delta$ so that $\exists V \in C$ so that $\{x_1, x_2\} \subset U$ but $f(V) = N_{\frac{\epsilon}{2}}(y)$ for some $y \in Y$ and $d_Y(f(x_1), y)$ and $d_Y(f(x_2), y) < \epsilon/2$ so that $d_Y(f(x_1), f(x_2)) < \epsilon$.

($\Longleftarrow$) Trivial ∎

Theorem 223 *Let (X, τ) be a compact topological space space and let $A \subset X$ such that $|A| = \infty$. Then, A has a limit point.*

Proof. Assume that A has no limit point. For each $x \in X$, there exists a neighborhood U_x of x such that $U_x \cap (A \backslash \{x\}) = \emptyset$. Let $C = \{U_x : x \in X\}$. Then, C is a cover of X so that we can have a finite subcover $\{U_{x_1}, U_{x_2}, ..., U_{x_n}\}$. Since $|A| = \infty$, we can have an i with $1 \leq i \leq n$ such that $|A \cap U_{x_i}| = \infty$ **(why does this not work for the finite case?)** but this contradicts the assumption that $U_{x_i} \cap (A \backslash \{x_i\}) = \emptyset$. ∎

Definition 224 *A topological space is* ***limit point compact*** *if every infinite subset has a limit point.*

Thus, by above, a compact metric space is limit point compact.

Definition 225 *Let (X,τ) be a compact topological space. X is said to be* ***sequentially compact*** *if every sequence in X has a convergent subsequence.*

Theorem 226 *If X is first countable topological space, then limit point compactness implies sequential compactness.*

Proof. Let $\{x_i\}$ be a sequence and let $A = \{x_i : i \in \mathbb{N}\}$. There are two cases. If $|A| < \infty$, then pick a constant subsequence. If $|A| = \infty$, then by limit point compactness, A has a limit point x. Let $\{U_i\}_{i=1}^{\infty}$ be a countable base at x. Assume that $U_{i+1} \subset U_i$. Let $i_1 = 1$ and for $j > 1$, let i_j be such that $i_j \geq i_{j-1}$ and such that $x_{i_j} \in U_j$.

We claim that $x_{i_j} \longrightarrow x$. Let U be a nieghborhood of x. Then, $\exists j \in \mathbb{N}$ so that $U_j \subset U$ but by construction, $x_{i_k} \in U_j \subset U$ if $k \geq j$. ■

Thus, compactness $\Longrightarrow$ limit point compact $\Longrightarrow$ sequentially compact (if space if first countable).

Theorem 227 *Let (X,d) be a metric space. If X is sequentially compact, then every open cover of X has a finite subcover.*

Proof. Assume that X is sequentially compact. We will prove that X satisfies the (a) Lebesgue number lemma and (b) show that for any $\epsilon > 0$, X can be covered by finitely many ϵ-balls. This would prove that X is compact because if we let C be a cover of X, $\delta > 0$ be a Lebesgue number for C and let $\epsilon = \delta/3$. There is a finite cover by ϵ-balls $\{N_\epsilon(x_1), ..., N_\epsilon(x_k)\}$. Each $N_\epsilon(x_i)$ has diameter $\frac{2}{3}\delta < \delta$ so that $N_\epsilon(x_i) \subset C_i \in C$ so that $\{C_1, ..., C_k\}$ is a finite subcover

For both of the assertions (a) and (b), we will work by proof by contradiction.

For (a), suppose that there is a cover C with no Lebesgue number. Thus, there is a sequence $\{C_n\}$ of sets not in C with $diam(C_n) < \frac{1}{n}$ so that C_i is not contained in any set of C. Let $x_n \in C_n$ and then $\{x_n\}$ is a sequence and hence has a convergent subsequence by hypothesis $\{x_{n_i}\}$ such that $x_{n_i} \longrightarrow x$ so that $x \in U$ for some $U \in C$. U is open so we can choose $\epsilon > 0$ such that $N_\epsilon(x) \subset U$. Assume that i is sufficiently large such that $\frac{1}{n_i} < \epsilon/2$ and $d(x_{n_i}, x) < \epsilon/2$. Then, if $y \in C_{n_i}$, then $d(y,x) \leq d(y, x_{n_i}) + d(x_{n_i}, x) \leq \frac{1}{n_i} + \frac{\epsilon}{2} < \epsilon$ so that $C_{n_i} \subset N_\epsilon(x) \subset U$. This contradicts choise of $\{C_n\}$ (we know that every body has a Lebesgue number)

For (b), assume that $\exists \epsilon > 0$ such that X cannot be covered by finitely many ϵ-balls. Pick $x_1 \in X$. Then, $N_\epsilon(x_1)$ does not cover X so we can pick $x_2 \in X \backslash N_\epsilon(x_1)$. By induction, pick $x_{n+1} \in X \backslash \left(\bigcup_{i=1}^{n} N_\epsilon(x_i) \right)$ so we can have a sequence $\{x_i\}$. We now show that $\{x_i\}$ has no convergent subsequence. Let $x \in X$. What if x is a limit of some subsequence of $\{x_i\}$? Let $U = N_{\frac{\epsilon}{2}}(x)$. By construction, $d(x_i, x_j) \geq \epsilon$ if $i \neq j$ so if $x_i \in N_{\frac{\epsilon}{2}}(x)$, then $x_j \notin N_{\frac{\epsilon}{2}}(x)$ (otherwise $d(x_i, x_j) \leq d(x_i, x) + d(x, x_j) < \epsilon$), a contradiction. ■

Thus, for a metric space, the notions of compactness, limit point compactness and sequential compactness are all equivalent.

7.3 Local Compactness

Not all spaces are compact. Sometimes we can embed a non-compact space in a nice compact space.

Definition 228 *Let (X, τ) be a topological space and $a \in X$. X is said to be locally compact at a if there exists a nieghborhood U of a and a compact set C such that $U \subset C$. X is locally compact if it locally compact at every point in X.*

The following is a motivational example.

Theorem 229 (Bolzano-Weirstrass) *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Proof. We can embed $\mathbb{R}$ into S^1 by stereographic projection ϕ. This map is continuous. Then, $\phi(\mathbb{R}) = S^1 \backslash \{N\}$ is homeomorphic to $\mathbb{R}$. Suppose we have a bounded subsequence $\{x_i\}$ in $\mathbb{R}$, with $\{x_i\}$ bounded by R. Then $\{\phi(x_i)\}$ lives in the closed set $\phi([-R, R])$. ■

Example 230 *Every compact space is locally compact.*

Example 231 $\mathbb{R}$ *is locally compact because $\forall r \in R$, $(r-\epsilon, r+\epsilon) \subset [r-\epsilon, r+\epsilon]$*

Example 232 $\mathbb{R}^n$ *is locally compact since $\forall r \in \mathbb{R}^n$, $r \in N_\epsilon(r) \subset \overline{N_\epsilon(r)}$*

Example 233 *If X has the discrete topology, then X is locally compact because for all $a \in X$, $a \in \{x\} \subset \{x\}$.*

Example 234 $\mathbb{Q}$ *is not locally compact: for $a \in \mathbb{Q}$, let U be a nieghborhood of a. Then, $U = \mathbb{Q} \cap I$ for some open interval I in $\mathbb{R}$. Suppose that $U \subset C$ where C is compact in $\mathbb{Q}$, so C is also compact in $\mathbb{R}$. So, C is closed in $\mathbb{R}$. Then, C has to contain irrational elements, which is a contradiction.*

What this says is that there aren't enough compact sets in $\mathbb{Q}$.. though there still are a lot.

Problem 235 *What are the compact subsets of $\mathbb{Q}$?*

Solution 236 *Clearly, all finite sets $F = \{q : q \in \mathbb{Q}\}$ are compact. Let K_0 be the collection of all finite subsets of $\mathbb{Q}$. Note that every finite union of members of K_0 is in K_0. This is our first collection and already countably infinite.*

Let q be any rational number and $\{q_n\}_{n=1}^{\infty}$ be a sequence of rational numbers such that $q_n \to q$. Then, $X = \{q_n : n \in \mathbb{N}\} \cup \{q\}$ is compact because by construction, every sequence in X has a convergent subsequence. There are various ways in which one could approach a rational number e.g., from left and right. Additionally, another possibility is, say for $q = 1$, then $q_n = 1 + \frac{k}{n} \to 1$ for each $k \in \mathbb{Q}$. Thus, for each q, there are possibly countably infinitely many ways in which we can have X. Let K_1 be the collection of all such X for each q. This is our second collection and again countably infinite.

Since the finite union of compact sets is compact, then $F\cup\bigcup_{j=1}^{m}X_j$ *is compact for each* $F\in K_0$ *and* $X_j\in K_1$*. Let* K_2 *be the collection of finite unions of elements of* K_1 *and* K_0*. This is our third collection of compact subsets of* $\mathbb{Q}$*. Again,* K_2 *is countably infinite.*

Now, for $Y\in K_2$*, for a fixed* n *and each* $q_n\in Y$*, let* $\{p_m\}_{m=1}^{\infty}$ *be a sequence of rational numbers such that* $\lim_{m\to\infty}p_m=q_n$*. Let* $Z=\{p_m: m\in\mathbb{N}\}\cup\{q_n\}$*. By construction,* Z *is compact. Essentially, for each member of the sequence* $\{q_n\}_{n=1}^{\infty}$*, we are letting* $\{p_m\}_{m=1}^{\infty}$ *be a sequence such that* $\lim_{m\to\infty}p_m=q_n$*. Let* K_3 *be the collection of all such sets.*

We can continue this way by adding more and more countably infinite collections of compacts subsets of $\mathbb{Q}$*.*

To show that the collection $K=\{K_0,K_1,...\}$ *consists of all compact subsets of* $\mathbb{Q}$*, assume* $\exists$ *compact subset* C *of* $\mathbb{Q}$ *such that* $C\neq K_i$ *for all* i*. Note that* $K_i\subset K_{i+1}$ *by construction. Hence each subset of* K *is a chain and is bounded above by the union of all elements of the chain. By Zorn's lemma,* K *has a maximal element, say* $C'=\bigcup_{i=0}^{\infty}K_i\in K$*. By Zorn's Lemma,* C' *is compact. Since* C *is compact, then* $C'\cup C$ *is compact. Also,* $C'\subset C'\cup C$*, a contradiction to the maximality of* C'*. Hence there does not exists such a* C*.*

Theorem 237 (Characterization of Locally Compact Hausdorff Spaces) *Let* (X,τ) *be a topological space and* $a\in X$*. Then,* X *is locally compact and Haudsorff if and only if there is a topological space* Y *such that*

1. *X is a subspace of Y*
2. $|Y\backslash X|=1$
3. *Y is compact, Hausdorff*

Furthermore, if Y *and* Y' *are two spaces satisfying* $1-3$*, then there is a homeomorphism* $h:Y\longrightarrow Y'$ *such that* $h(x)=x$ *for all* $x\in X$

Proof. $(\Longrightarrow)$

We will first prove uniqueness upto homeomorphism. Let Y,Y' be spaces satisfying $1-3$. Let $Y\backslash X=\{\infty\}$ and $Y'\backslash X=\{\infty'\}$. Define $h:Y\longrightarrow Y'$ given by

$$h(x)=\begin{cases} x & \text{if } x\in X\\ \infty' & \text{if } x=\infty\end{cases}$$

Then, h is bijective. If $U\subset Y$ is open, then $h(U)\subset Y'$ is open. By symmetry, h^{-1} is also an open map. So we simply need to prove that $h(U)$ is open in Y' for open U. If $\infty\notin U$, then clearly $h(U)$ is open. If $\infty\in U$, then let $C=Y\backslash U$. This set closed, a subset of a compact space Y and therefore compact. Thus, $h(C)=C$ is compact. Since Y' is Hausdorff, C is closed in Y' and so $h(U)=Y'\backslash C$ and is open.

Now, let $Y = X \cup \{\infty\}$. Then, 2 holds and X is a subset of Y. Let $\tau_1 = \{U : U \text{ is open in } X\}$ and $\tau_2 = \{Y \backslash C : C \text{ is compact in } X\}$. Then, $\tau_1 \cup \tau_2$ is a compact topological space on Y: $\emptyset$ is open in X so that $\emptyset \in \tau_1$ and $\emptyset$ is compact in X so that $Y \in \tau_2$. For intersection, there are three cases. If both open sets are open in X, then their intersection is clearly in τ_1. If both open sets are in τ_2, then $U_1 = Y \backslash C_1$ and $U_2 = Y \backslash C_2$ for some compact sets C_1, C_2 in X. Then, $U_1 \cap U_2 = Y \backslash (C_1 \cup C_2) \in \tau_2$. If $U_1 \in \tau_1$ and $U_2 \in \tau_2$, then $U_2 = Y \backslash C_2$ and $U_1 \cap U_2 = X \backslash C_2$ which is open because C_2 is closed since X is Hausdorff. Finally, let $\{U_\alpha\}_{\alpha \in A}$ be a collection of elements in τ. If $A = B \sqcup C$ where $U_\alpha \in \tau_1$ is $\alpha \in B$ and $U_\alpha \in \tau_2$ is $\alpha \in C$. For each $\alpha \in C$, $\exists C_\alpha$ compact in X so that $U_\alpha = Y \backslash C_\alpha$ so that $\bigcup_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in B} U_\alpha \cup \bigcup_{\alpha \in C} U_\alpha$
$= \bigcup_{\alpha \in B} U_\alpha \cup \bigcup_{\alpha \in C} (Y \backslash C_\alpha)$
$= \bigcup_{\alpha \in B} U_\alpha \cup \left(Y \backslash \bigcap_{\alpha \in C} C_\alpha \right)$
$= U \cup (Y \backslash C) = Y \backslash (C \backslash U)$. This is open because $C \backslash U$ is closed because it is compact.

To show 1, we need to check that $U \subset X \iff U = V \cap X$ for $V \in \tau$. If $V \in \tau_1$, then $V \cap X = V$ is open in X. If $V \in \tau_2$, then $V = Y \backslash C$ for C compact in X. Then, $X \cap (Y \backslash C) = X \backslash C$ where C is closed and so $X \backslash C$ is open in X.

Let $\mathcal{A}$ be an open cover of Y. There is at least 1 set $U_1 = Y \backslash C$ for C compact in X. Let $\mathcal{A}_1 = \mathcal{A} \backslash \{U_1\}$. Then, $\mathcal{A}_1$ is an open cover of C but C is compact and so it can be covered by finitely many sets of the form $\{D \cap X : D \in \mathcal{A}_i\}$. The corresponding sets in $\mathcal{A}_1$, along with U_1, are a cover of Y.

Let x, y be distinct points in Y. If $x, y \in X$, then we can separate them since X is Hausdorff. If $y = \infty$. Then pick a niehgborhood U of x and a compact C in X such that $x \in U \subset C$ and $V = Y \backslash C$. Then, U, V seperate x and ∞.

($\Longleftarrow$) Assume that Y exists for X and $1-3$ holds. Then, X is Hausdorff because a subspace of a Hausdorff space is Hausdorff. Also, it is locally compact because for $x \in X$, we can find disjoint neighborhoods U, V of x and ∞ so $V = Y \backslash C$ where C is compact in X and $x \in U \subset C$. ∎

8 Countability and Separability Axioms

As a motivation, recall that metric spaces have many nice properties: they are first countable and Hausdorff, for instance. However, not all topological spaces are metric spaces. So, if we are given a topological space, how can we tell that it is a metric space? Another way to ask this question is to determine which sets in a topological space can be distinguished by continuous functions: let $A, B \subset X$ be disjoint subsets of A. Is there a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ if $x = B$. Not in general, there isn't. Consider $A = (0,1)$ and $B = (1,2)$ and $x_n = 1 - \frac{1}{n}$ and $y_n = 1 + \frac{1}{n}$. Then, $f(x_n) \to 1$ and $f(y_n) \to 1$ but $x_n \in A$ and $y_n \in B$. Should we require A and

B to be closed? Then the function $f(x) = \frac{d(x,A)}{d(x,A)+d(x,B)}$ works: since A and B are disjoint, the denominator is never zero. If $x \in A$, then $f(x) = 0$ whereas if $x \in B$, then $f(x) = 1$.

8.1 Countability Properties

Recall the following for (X, τ) a topological space:

1. X is first countable if X has a countable local base at each point
2. X is second countable if X has a countable basis.

For example, for any $n \in \mathbb{N}$, $\mathbb{R}^n$ is both first and second countable. On the other hand, think about $\mathbb{R}^{\mathbb{N}}$ in the product topology. $\mathbb{R}^{\mathbb{N}}$ is second countable and hence first countable with basis

$$B = \left\{ \prod_n U_n : U_n = (a_n, b_n) \text{ with } a_n, b_n \in \mathbb{Q} \text{ and } U_n = \mathbb{R} \text{ for all but finitely many } n \right\}$$

However, with the uniform topology stemming from metric $D(x,y) = \sup_k d(\pi_k(x), \pi_k(y))$ is first countable but not second countable. To see this, we can use the following:

Lemma 238 *If X is second countable and $A \subset X$ is a discrete subspace. Then, A is countable.*

Proof. Let $\mathcal{B}$ be a countable basis for X. For each $a \in A$, there exists $B_a \in \mathcal{B}$ such that $B_a \cap A = \{a\}$. Consider the function $f : A \longrightarrow \mathcal{B}$ with $f(a) = B_a$. By construction, if $a, b \in A$ are distinct, then $B_a \neq B_b$ and so f is injective hence A is countable. ■

Now, to show that the $\mathbb{R}^{\mathbb{N}}$ with the uniform topology is not second countable, consider $A = \left\{x \in \mathbb{R}^{\mathbb{N}} : \pi_k(x) \in \{0,1\} \, \forall k \in \mathbb{N}\right\}$. That is, A is the set of sequences which contain only zero or one. If $x, y \in A$ are distinct, then $\exists k$ such that $\pi_k(x) \neq \pi_k(y)$ so that $D(x,y) = 1$. Hence A has the discrete topology. To show that A is uncountable, assume the contrary: then $A = \{a_1, a_2, ...\}$. Let $p \in A$ such that $\pi_k(p) \neq \pi_k(a_k) \implies p \notin A$

Theorem 239 *Let (X, τ) be a topological space.*

1. *If X is first countable and $B \subset X$, then B is first countable*
2. *If X is second countable and $B \subset X$, then B is second countable*
3. *If C is countable and $\{X_i\}_{i \in C}$ is a collection of first countable topological spaces, then $\prod_{i \in C} X_i$ is first countable*
4. *If C is countable and $\{X_i\}_{i \in C}$ is a collection of second countable topological spaces, then $\prod_{i \in C} X_i$ is second countable*

Only proofs for second countable will be demonstrated. The other will follow

Proof. (2) Let D be a countable basis for X and let $B \subset X$ and $D' = \{U \cap B : U \in D\}$. We now show that D' is a basis: if $a \in B$, then $\exists U \in D$ such that $a \in D$ and so $a \in U \cap B$. Suppose $e_1, e_2 \in D$ and $a \in e_1 \cap e_2$. Then $\exists d_1, d_2 \in D$ such that $e_1 = d_1 \cap B$ and $e_2 = d_2 \cap B$ and so $a \in d_1 \cap d_2 \implies \exists d_3 \in D$ such that $a \in d_3 \subset d_1 \cap d_2$ and so if $e_3 = d_3 \cap B$ so that $a \in e_3 \subset e_1 \cap e_2$.

(4) Let F_i be a countable basis for X_i for each $i \in C$ and let

$$F = \left\{ \prod_{i \in C} U_i : U_i \in F_i \text{ and } U_i = X_i \text{ for all but finitely many } i \right\}$$

We first need to check that every element of F is open in the product topology on $\prod_{i \in C} X_i$. Clearly, if $y \in \prod_{i \in C} X_i$, then $\exists f \in F$ such that $y \in f$ with $f = U_1 \times X_2 \times X_3 \times \ldots$. Let $f_1, f_2 \in F$. Then, $f_1 \cap f_2$ is of the form $\prod_{i \in C} U_1^i \cap U_2^i$ where $U_1^i, U_2^i \in F_i$ and $U_1^i \cap U_2^i = X_i$ for all but finitely many i. If $y \in f_1 \cap f_2$, then for each i, there is $U_3^i \in F_i$ (or $U_3^i = X_i$) such that $\pi_i(y) \in U_i^3 \subset U_i^1 \cap U_i^2$. Therefore, $y \in \prod_{i \in C} U_3^i \subset \prod_{i \in C} U_1^i \cap U_2^i = f_1 \cap f_2$. F is countable since F_i and C are countable. $\blacksquare$

Theorem 240 *Let X be a second countable topological space.*

1. *If C is an open cover of X, then C contains a countable subcover (Lindelöf)*
2. *X is separable*

Proof. (1) Let $\{B_n\}$ be a countable basis for X and let C be an open cover of X. Define $A = \{n \in \mathbb{N} : \exists U \in C \text{ such that } B_n \subset U\}$. For each $n \in A$, let $C_n \in C$ be a set such that $B_n \subset C_n$. We need to show that $\{C_n : n \in A\}$ is a cover. $\{C_n : n \in A\}$ is certainly countable. Let $x \in X$. Choose $U \in C$ such that $x \in U$. Since U is open, $\exists n \in \mathbb{N}$ such that $x \in B_n \subset U$, so $n \in A$ and hence $x \in B_n \subset C_n$ so $\{C_n\}$ is a cover.

(2) Let $x_n \in B_n$. Let $D = \{x_n\}$. This is a dense subset. To show this, consider $y \in X$ and $U \ni y$ be a neighborhood of y. Then, $\exists n$ such that $x \in B_n \subset U$ and so $x_n \in U$. So, each neighborhood of y intersects D, so $y \in \overline{D}$. Since y was arbitrary, $X = \overline{D}$ $\blacksquare$

Neither one of these imply that X is second countable. As an example, consider $\mathbb{R}_l$, the real numbers with the lower limit topology with $B = \{[a, b) : a, b \in \mathbb{R}\}$. Then, $\mathbb{R}_l$ is first countable: let $x \in \mathbb{R}_l$. Consider $\left\{[x, x + \frac{1}{n}) : n \in \mathbb{N}\right\}$. It is easy to see that for any basis, the set $\left\{[x, x + \frac{1}{n}) : n \in \mathbb{N}\right\}$ is a local basis. However, $\mathbb{R}_l$ is not second countable. Suppose that B' is a basis for $\mathbb{R}_l$. For each $x \in \mathbb{R}_l$, we know that $\exists B_x \in B'$ such that $x \in B_x \subset [x, x + 1)$. If we can show that for different reals x, y, $B_x \neq B_y$, holds. $x \in B_x$ but $x \notin B_y$. $\mathbb{R}_l$ is separable since $\mathbb{Q}$ is dense in $\mathbb{R}$. It is also Lindelöf.

8.2 Separability Axioms

Definition 241 *If (X, τ) is a T_1 space, then X is called **regular** if for each $a \in X$ and each $B \subset X$ closed and not containing a, there are disjoint open sets U and V such that $a \in U$ and $B \subset V$.*

Definition 242 *If (X, τ) is a T_1 space, then X is called **normal** if for every $A, B \subset X$ closed and disjoint, there are disjoint open sets U and V such that $A \subset U$ and $B \subset V$.*

Theorem 243 *Let (X, τ) be a T_1 space. Then,*

1. *X is regular $\iff$ for each $a \in X$ and each neighborhood U containing a, there is a neighborhood V of a containing a such that $a \in \overline{V} \subset U$.*

2. *X is normal $\iff$ for each closed $A \subset X$ and each open set U with $A \subset U$, there is an open set $V \supset A$ such that $A \subset \overline{V} \subset U$.*

Proof. (1 $\Longrightarrow$)

Let X be regular. Let $a \in X$ and let U with $a \in U$ be a neighborhood of a. Let $B = U^c$. Then, B is closed and disjoint from a and so we can find disjoint open sets V and W such that $a \in V$ and $B \subset W$. Let $y \in B$. Then, W is a neighborhood of y disjoint from V and so $y \notin \overline{V}$ so $B \cap \overline{V} = \emptyset$ and that $\overline{V} \subset U$

(1 $\Longleftarrow$). Let $a \in X$ and $B \subset X$ be closed not containing a. Let $U = B^c$ be an open set containing a. So there is a neighborhood V of a with $a \in V$ such that $\overline{V} \subset U$. Let $W = \overline{V}^c$. It is easy to see that V and W are disjoint open sets so that $a \in V$ and $B \subset W$. ■

Theorem 244

1. *A subspace of a Hausdorff space is Hausdorff*

2. *A subspace of a regular space is regular*

3. *Product of Hausdorff Spaces is Hausdorff*

4. *Product of regular spaces is regular*

Proof. (1) If X is Hausdorff and $B \subset X$ and $a, b \in B$ are distinct points, then there are disjoint open (in X) U, V such that $a \in U$ and $b \in V$ but then if we let $U' = B \cap U$ and $V' = B \cap V$, then these are disjoint open (in B) sets containing a, b respectively.

(2) Let $B \subset X$. So, B is T_1. Let $a \in B$ and $C \subset B$ be a closed (in B) set not containing a. Then, $\overline{C} \cap B = C$ and so $a \notin \overline{C}$. So, these are open (in X) disjoint sets U, V so that $a \in U$ and $\overline{C} \subset V$. Then, $U' = B \cap U$ and $V' = B \cap V$ and so $a \in U'$ and $C \subset V'$ sets are disjoint and open.

(3) Let $u, v \in \prod_{\alpha \in A} X_\alpha$ be distinct points. That, if $u = (u_\alpha)_{\alpha \in A}$ and $v = (v_\alpha)_{\alpha \in A}$, then $\exists \beta \in A$ such that $u_\beta \neq v_\beta$. Since X_β is Hausdorff, $\exists U_\beta$ and V_β open in X_β, disjoint with $u_\beta \in U_\beta$ and $v_\beta \in V_\beta$. Let $U = \pi_\beta^{-1}(U_\beta)$ and $V = \pi_\beta^{-1}(V_\beta)$.

(4) Let $\{X_\alpha\}_{\alpha\in A}$ be a family of regular saces. X is Hausdorff so X is T_1. Let $x = (x_\alpha)_{\alpha\in A} \in X$ and let U be a neighborhood of x. Choose a basis element of form $\prod_{\alpha\in A} U_\alpha$ such that $x \in \prod_{\alpha\in A} U_\alpha \subset U$. Since $x_\alpha \in U_\alpha$, we can find a neighborhood V_α of x_α such that $x \in \overline{V_\alpha} \subset U_\alpha$. If $U_\alpha = X_\alpha$, then $V_\alpha = X_\alpha$, then $V = \prod_{\alpha\in A} V_\alpha$ is open in X and $\overline{V} = \prod_{\alpha\in A} \overline{V_\alpha}$ and so $x \in \overline{V} \subset U$ so X is regular. ■

Example 245 *$\mathbb{R}_l$, reals with lower limit topology, has basis $\{[a,b) : a,b \in \mathbb{R}\}$. Then, $\mathbb{R}_l$ is normal and hence regular and Hausdorff. $\mathbb{R}_l$ is T_1 since $\mathbb{R}$ is T_1 with standard topology and $\mathbb{R}_l$ is finer. Let $A, B \subset \mathbb{R}_l$ be disjoint closed sets. For each $a \in A$, there is a basis element $[a, x_a)$ that is disjoint from B. Let $U = \bigcup_{a\in A} [a, x_a)$. Then, $A \subset U$. Repeat construction for B to get $V \supset B$. If $U \cap V \neq \emptyset$, then $\exists a \in A, b \in B$ such that $[a, x_a) \cap [b, x_b) \neq \emptyset$ so that $a \in [b, x_b)$ and $b \in [a, x_a)$, an impossibility.*

Example 246 *$\mathbb{R}_l^2$ is regular but not normal, called the Sorgenfrey Plane. Suppose that $\mathbb{R}_l^2$ is normal. Let $L = \{(x, -x) : x \in \mathbb{R}_l\}$. This is just a diagonal line $y = -x$. Because $\mathbb{R}_l$ is Hausdorff, L is closed in $\mathbb{R}_l^2$. L is a discrete subspace of $\mathbb{R}_l^2$. Let $A \subset L$ and so $L \backslash A$ is closed in L. So, $L \backslash A$ is also closed in $\mathbb{R}_l^2$. For each $A \subset L$, let U_A and V_A be disjoint open sets such that $A \subset U_A$ and $L \backslash A \subset V_A$. Let $D = \mathbb{Q}^2$. D is dense in $\mathbb{R}_l^2$. Define a map $\theta : 2^L \longrightarrow 2^D$ by $\theta(A) = D \cap U_A$, $\theta(\emptyset) = \emptyset$ and $\theta(L) = D$. This map is injective: let $A \subset L$ be a proper subset. $\theta(A)$ is neither empty nor D because D dense and so $D \cap U_A \neq \emptyset$ and $D \cap V_A \neq \emptyset$. Let $B \subset L$ be a proper subset with $A \neq B$. WLOG, assume $x \in A$ but $x \notin B$. Then, $x \in L \backslash B \implies x \in V_B$ and so $x \in U_A \cap V_B$. This set is open so $\exists c \in D \cap (U_A \cap V_B)$ so $c \in D \cap U_A$ but $c \notin D \cap U_B$. Hence θ is injective. Thus, we can find an injective map $\psi : 2^D \longrightarrow L$ and $\psi \circ \theta = 2^L \longrightarrow L$ is injective, a contradiction. The injective map ψ is possible since D is countable and L is uncountable. The injective map ψ may be defined on for $E \subset \mathbb{N}$ by $E \longmapsto \Sigma \frac{a_i}{10^i}$ with $a_i = 0$ if $i \notin E$ and 1 otherwise.*

Theorem 247 *Let (X, τ) be a topological space such that X is regular and second countable, then X is normal.*

Proof. Let X be regular with countable basis C. Let $A, B \subset X$ be closed and disjoint. For each $a \in A$, we can choose a neighborhood U of a and is disjoint from B. Since X is regular, there is a neighborhood $V \ni a$ such that $\overline{V} \subset U$. We can choose a basis element $D \in C$ such that $a \in D \subset V$ and so we can let $\{D_n\}$ be the collection of basis elements constructed as above. $\{D_n\}$ is an open cover of A whose closures are disjoint from B. Similarly, we can construct $\{E_n\}$ of B. Let $\mathcal{D} = \bigcup D_n$ and $\mathcal{E} = \bigcup E_n$ so $\mathcal{D}$ and $\mathcal{E}$ are open sets that contain A and B respectively. There may exist k such that $D_k \cap E_k \neq \emptyset$. For each n, let $D_n' = D_n \backslash \bigcup_{i=1}^{n} \overline{E}_i$, $E_n' = E_n \backslash \bigcup_{i=1}^{n} \overline{D}_i$, $D' = \bigcup D_n'$ and $E' = \bigcup E_n'$. Let $a \in A$.

Then, $\exists n \in \mathbb{N}$ such that $a \in D_n$ but for each i, $1 \leq i \leq n$, we know that $\overline{E}_i$ is disjoint from A hence the new collection is still an open cover of A. ■

Theorem 248 *Let (X, τ) be a metrizable topological space, then X is normal.*

Proof. If $A, B \subset X$ are closed and disjoint, we can define a continuous function $f(x) = \frac{d(x,A)}{d(x,A)+d(x,B)}$. The sets $U = f^{-1}\left(\left(-\infty, \frac{1}{3}\right)\right)$ and $V = f^{-1}\left(\left(\frac{2}{3}, \infty\right)\right)$ give us our required separation. ■

Theorem 249 *Let (X, τ) be a compact, Hausdorff topological space. Then, X is normal.*

Proof. If $a \in X$ and $B \subset X$ is closed not containing a. Then, B is compact and we can separate it from a by open sets. Thus, X is regular. Now for A and B closed and disjoint, for each $a \in A$, define U_a and V_a with open and disjoint sets $a \in U_a$ and $B \subset V_a$. Notice that $\{U_a\}_{a \in A}$ is a cover for A so we can find $\{U_{u_1}, ..., U_{u_n}\}$ covering A and $\{V_{a_1}, ..., V_{a_n}\}$. The union of these sets and the union of $\{U_{u_1}, ..., U_{u_n}\}$ and intersection of $\{V_{a_1}, ..., V_{a_n}\}$ gives us the required sets. ■

Theorem 250 (Urhysohn Metrization Theorem) *If X is regular and second countable, then X is metrizable.*

It does not follow that a normal space is metrizable.

To prove this, we will need what's called the Urhysohn's (Urj-jon) Lemma:

Theorem 251 *If X is normal, $A, B \subset X$ are closed and disjoint sets and $[a, b] \subset \mathbb{R}$ is an interval, then there is continuous function $f : X \longrightarrow [a, b]$ so that $f(A) = \{a\}$, $f(B) = \{b\}$*

Proof. It suffices to work with the closed interval $[0, 1]$. Let $P = \mathbb{Q} \cap [0, 1]$. Define, for each $p \in P$, an open set U_p so that if $q \in P$ and $p < q$, then $\overline{U_p} \subset U_q$. We will need to use induction.

For the base case, since P is countable, we can have $P = \{p_1, ...\}$ with $p_1 = 1$ and $p_2 = 0$. Let $U_{p_1} = U_1 = X \backslash B$. Next, since A is closed and $A \subset U_1$, we can use normality of X to find U_0 such that $A \subset U_0 \subset \overline{U_0} \subset U_1$.

Let $P_n = \{p_1, ..., p_n\}$. Assume that we have defined U_p for each $p \in P_n$. We need to define $U_{p_{n+1}}$. Since $P_{n+1} = P_n \cup \{p_{n+1}\}$ is a totally ordered subset of $\mathbb{Q}$, every element of P_{n+1} has an immediate predeccessor and immediate successor other than the smallest element of P_{n+1}. We can let p and q be the immediate predeccessor and successor respectively of p_{n+1}. By construction, $\overline{U_p} \subset U_q$ and by normality, $\exists U_{p_{n+1}}$ (open) such that $\overline{U_p} \subset U_{p_{n+1}} \subset \overline{U_{p_{n+1}}} \subset U_q$.

To complete the inductive process, we need to show that if $s, t \in P_{n+1}$ and $s < t$, then $\overline{U_s} \subset U_t$. If $s, t \in P_n$, then the statement holds by default. If $s = p_{n+1}$, then $t \in P_n$ and $t \geq q$ and so $\overline{U_s} \subset \overline{U_q} \subset U_t$. Similarly, if $t = p_{n+1}$, similar argument works.

Thus, if $s, t \in P$, then $\exists n$ such that $s, t \in P_n$ so $\overline{U_s} \subset U_t$.

As a formality, we will let $U_p = \emptyset$ if $p < 0$ and $U_p = X$ if $p > 1$.

Next, for each $x \in X$, define $Q(x) = \{p \in \mathbb{Q} : x \in U_p\}$. By construction, $Q(x)$ is non-empty and $Q(x) \subset (0, \infty)$. Define $f : X \longrightarrow [0,1]$ by $f(x) = \inf Q(x)$. If $a \in A$, then $A \subset U_0$ and so $0 \in Q(a)$ and so $\inf Q(a) = 0$ and so $f(a) = 0$. If $b \in B \subset X \backslash U_1$ and $U_p \subset \overline{U_p} \subset U_1$ for all $p \leq 1$ and $B \subset X \backslash U_p$ for all $p \leq 1$ so $p \notin Q(b) = (1, \infty)$ for all $p < 1$ and so $f(b) = 1$.

We now need to show f is continuous. This will follow if f satisfies the following:

1. If $x \in \overline{U_r}$, then $f(x) \leq r$. Equivalently, if $f(x) > r$ then $x \notin \overline{U_r}$
2. If $x \notin U_r$, then $f(x) \geq r$. Equivalently, if $f(x) < r$, then $x \in U_r$

To see this, let $a \in X$ and let $(c,d) \subset \mathbb{R}$ such that $f(a) \in (c,d)$. We will find a neighborhood U of a such that $f(U) \subset (c,d)$. Pick $p, q \in \mathbb{Q}$ with the property that $c < p < f(a) < q < d$. Let $U = U_q \backslash \overline{U_p}$ be an open set. Since $f(a) < q$, then $a \in U_q$. Also, $p < f(a)$, so that $a \notin \overline{U_p}$. Let $d \in U$. Then, $d \in U_q \subset \overline{U_p}$ and $f(d) \leq q < d$ and $d \in \overline{U_p}$ so $f(d) \geq p < c$ and so $f(d) \in (c,d)$.

Now, to prove 1, if $x \in \overline{U_r}$, then $x \in U_s$ for $s > r$ and so $(r, \infty) \subset Q(x)$ and so $f(x) \leq r$. For 2, if $x \notin U_r$, then $x \notin U_s$ for $s < r$ and so $f(x) \geq r$. ■

This does not hold for regular spaces: that is, in a regular space, you can't separate using continuous functions. For this, we need a few more definitions:

Definition 252 *If X is a T_1 space, then X is* ***completely regular*** *if for each $a \in X$ and each $B \subset X$ closed set not containing a, there is a continuous function $f : X \longrightarrow [0,1]$ so that $f(a) = 0$ and $f(B) = \{1\}$.*

So, a normal space is completely regular, an immediate consequence of Urhysohn's lemma and thus is regular.

Theorem 253 *If X is completely regular and $B \subset X$ is a subspace, then B is completely regular.*

Proof. Let $a \in B$ and $C \subset B$ be closed such that $a \notin C$. Then, $C = \overline{C} \cap B$ and $a \notin \overline{C}$. Since X is completely regular, there is a continuous function $f : X \longrightarrow [0,1]$ such that $f(a) = 0$ and $f\left(\overline{C}\right) = \{1\}$. Let $f|_B : B \longrightarrow [0,1]$. Then, $f|_B$ is continuous and $f|_B(a) = f(a) = 0$ and $f|_B(C) = f|_B\left(\overline{C} \cap B\right) = f\left(\overline{C}\right) = \{1\}$. ■

Theorem 254 *If $\{X_\alpha\}_{\alpha \in A}$ is a collection of completely regular spaces, then the product in the product topology is completely regular.*

Proof. Let $B \subset X = \prod\limits_{\alpha \in A} X_\alpha$ be closed such that $x \notin B$. Choose a basis element $\prod\limits_{\alpha \in A} U_\alpha$ with $U_\alpha = X_\alpha$ for all but finitely many A so that $x \in \prod\limits_{\alpha \in A} U_\alpha$ and $\prod\limits_{\alpha \in A} U_\alpha$ is disjoint from B. Let $F = \{\alpha : U_\alpha \neq X_\alpha\}$. For each $\alpha \in F$, choose continuous $f|_{\alpha} : X_{\alpha} \longrightarrow [0,1]$ so that $f_{\alpha}(x_{\alpha}) = 1$ and $f_{\alpha}(X_{\alpha} \backslash U_{\alpha}) = \{0\}$. Let $\varphi_{\alpha} : X \longrightarrow [0,1]$ be given by $f_{\alpha} \circ \pi_{\alpha}$. Define $f(x) = \prod_{\alpha \in F} \varphi_{\alpha}(x) = \prod_{\alpha \in F} f_{\alpha} \circ \pi_{\alpha}(x) = \prod_{\alpha \in F} f_{\alpha}(x_{\alpha}) = 1$. Observe that φ_{α} vanishes outside of $\pi_{\alpha}^{-1}(U_{\alpha})$ and $\prod_{\alpha \in A} U_{\alpha} = \bigcap_{\alpha \in F} \pi_{\alpha}^{-1}(U_{\alpha})$ and so f vanishes outside $\prod_{\alpha \in A} U_{\alpha}$ so $f(B) = \{0\}$. ■

Theorem 255 (Urhyson's Metrization Theorem) *If X is regular and second countable, then X is metrizable*

If X is a topological space and $A \subset X$ is a subspace and $f : A \longrightarrow \mathbb{R}$ is continuous, can you extend f continuously to X? Not always. Consider $A = (0, \infty)$ and $f(x) = \frac{1}{x}$. This cannot be extended continuously to 0. Things change a little if A is closed but not quite. There is a cool proof with a good mix of calculus and topology called Tietza Extension Theorem.

For its proof, we need to understand convergence of a sequence of functions in a topological setting. Consider a sequence $\{f_n\}_{n=1}^{\infty}$ of functions from X to $\mathbb{R}$. For now, we do not need a topology on X. If we have another function $f : X \longrightarrow \mathbb{R}$, then $\{f_n\}_{n=1}^{\infty}$ is said to converge to f point-wise if $\lim_{n \to \infty} f_n(x) = f(x)$ for each $x \in X$ **(check quantifier placement)**. This is the weakest possible notion of convergence.

If $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions (i.e. X has a topology) and converges pointwise to f, is f continuous? Not necessarily. Let $X = [0,1]$ and $f_n : [0,1] \longrightarrow [0,1]$ with $f_n(x) = x^n$. The function $f : X \longrightarrow [0,1]$ defined by $f(x) = 0$ if $x \in [0,1)$ and 1 otherwise is the point-wise limit of $\{f_n\}_{n=1}^{\infty}$. This, point-wise convergence takes away continuity of the limit. The problem here is that for each number x, we have to go extremely far out in n. This indicates that for finding that N in convergence is independent of x.

If we want the limits to be continuous as well, we could define convergence of $\{f_n\}_{n=1}^{\infty}$ to f uniformly if for all x, $\lim_{n \to \infty} f_n(x) = f(x)$. The N does not depend on x. Inherent in the proof is the use of a metric for uniform convergence!

Theorem 256 *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions and $f : X \longrightarrow \mathbb{R}$ is a function such that for each x, $\lim_{n \to \infty} f_n(x) = f(x)$, then f is continuous.*

Proof. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ so that $|f_n(x) - f(x)| < \frac{\epsilon}{3}$ for $n \geq N$. Let $m \geq N$ and let U be a neighborhood of x such that if $y \in U$, then $|f_m(x) - f_m(y)| < \frac{\epsilon}{3}$. Now, for $y \in U$ so that $|f(x) - f(y)| \leq |f(x) - f_m(x)| + |f_m(x) - f_m(y)| + |f_m(y) - f(y)| < \epsilon$ ■

Now, consider $f : A \longrightarrow [-r, r]$ continuous with A closed in a topological space X. If f is continuous, then we can define a continuous $g : X \longrightarrow \mathbb{R}$

such that

1. (a) $|g(x)| < \frac{1}{3}r$ (says how big g is allowed to be)

(b) $|g(a) - f(a)| \leq \frac{2}{3}r$ for all $a \in A$ (says distance of points from g and f are not too big, either, implying that g is relatively close to f)

Proof. To show this, take the interval $[-r, r]$ and chop it into three equally-sized closed intervals $I_1 = \left[-r, -\frac{1}{3}r\right]$, $I_2 = \left[-\frac{1}{3}r, \frac{1}{3}r\right]$ and $I_3 = \left[\frac{1}{3}r, r\right]$. Thenm $B = f^{-1}(I_1)$ and $C = f^{-1}(I_3)$ are closed in A and hence closed and disjoint subsets in X. We can then yse Urhysohn's lemma to construct $g : X \longrightarrow \left[-\frac{1}{3}r, \frac{1}{3}r\right]$ so that $g(B) = \left\{-\frac{1}{3}r\right\}$ and $g(C) = \left\{\frac{1}{3}r\right\}$. This satisfies (a).

For (b), we need to consider three cases. If $a \in B$, then $f(a) \in I_1$ and $g(a) = -\frac{1}{3}r$. If $a \in C$, then $f(a) \in I_3$ and $g(a) = \frac{1}{3}r$. If $a \notin B$ and $a \notin C$, then $f(a) \in I_2$ and, by construction, $g(a) \in I_2$. ■

Theorem 257 (Tietza Extension Theorem) *If (X, τ) is a normal topological space and A is a closed in X, then*

1. *If $f : A \longrightarrow [a, b]$ is continuous, then there is a continuous extension $\overline{f} : X \longrightarrow [a, b]$ of f*

2. *If $f : A \longrightarrow (a, b)$ is continuous, then there is a continuous extension $\overline{f} : X \longrightarrow (a, b)$ of f*

Proof. (1) We will prove for $[-1, 1]$. For $f : A \longrightarrow [-1, 1]$, we can use Urhyson's lemma to construct $g_1 : X \longrightarrow \left[-\frac{1}{3}, \frac{1}{3}\right]$ as above so that $|f(a) - g_1(a)| < \frac{2}{3}$ for each $a \in A$. Apply claim to $f - g_1$ which is a function from A to $\left[-\frac{2}{3}, \frac{2}{3}\right]$ to construct $g_2 : X \longrightarrow \mathbb{R}$ so that $|g_2(x)| < \frac{1}{3}\frac{2}{3}$ and $|(f - g_1)(a) - g_2(a)| < \frac{2}{3}\frac{2}{3}...$ and construct $g_{n+1} : X \longrightarrow \mathbb{R}$ so that $|g_{n+1}(x)| < \frac{1}{3}\left(\frac{2}{3}\right)^n$ and $|f(a) - g_1(a) - ... - g_{n+1}(a)| < \left(\frac{2}{3}\right)^{n+1}$. That is, we have a sequence $\{g_i : X \longrightarrow [-1, 1]\}$. Let $g : X \longrightarrow \mathbb{R}$ be given by $g(x) = \sum_{i=1}^{\infty} g_i(x)$ so that $|g(x)| \leq \sum_{i=1}^{\infty} |g_i(x)| \leq \frac{1}{3}\sum_{i=1}^{\infty}\left(\frac{2}{3}\right)^{i-1} = \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = \frac{1}{3}\left(\frac{1}{\frac{1}{3}}\right) = 1$ so $g : X \longrightarrow [-1, 1]$ is well-defined. We now show that g is continuous. Let $s_n(x) = \sum_{i=1}^{n} g_i(x)$. Then, $s_n \longrightarrow g$ point-wise. Let $k > n$ where $k, n \in \mathbb{N}$. Then, $|s_k(x) - s_n(x)| = \left|\sum_{i=n+1}^{k} g_i(x)\right| \leq \sum_{i=n+1}^{k} |g_i(x)| < \sum_{i=n+1}^{k}\left(\frac{2}{3}\right)^{i-1} < \sum_{i=n+1}^{\infty}\left(\frac{2}{3}\right)^{i-1} = \left(\frac{2}{3}\right)^n$. Thus, $|g(x) - s_n(x)| < \left(\frac{2}{3}\right)^n$. Now let $\epsilon > 0$. Pick N such that $\left(\frac{2}{3}\right)^N < \epsilon$ so that if $n \geq N$, then $|g(x) - s_n(x)| < \left(\frac{2}{3}\right)^N < \epsilon$ independent of x so that $s_n \longrightarrow g$ uniformly. Since each s_n are continuous, then g is also continuous. Since $|f(a) - s_n(a)| < \left(\frac{2}{3}\right)^n$ by construction and $s_n(a) \longrightarrow g(a)$, then $|f(a) - g(a)| < \left(\frac{2}{3}\right)^n$ for any n so $g(a) = f(a)$

(2) Since $(-1, 1) \subset [-1, 1]$, we can find $g : X \longrightarrow [-1, 1]$ so that $g(a) = f(a)$ for any $a \in A$. Let $D = g^{-1}(\{-1, 1\})$. Since $\{-1, 1\}$ is closed in $[-1, 1]$, then D is closed in X. D is disjoint from A (since $g(a) = f(a) \in (-1, 1)$). Use

Urhyson's lemma to find continuous function $h : X \longrightarrow [0,1]$ such that $h(D) = \{0\}$ and $h(A) = \{1\}$. Let $\overline{f} : X \longrightarrow (-1,1)$ be given by $\overline{f}(x) = h(x)\,g(x)$. This function is continuous as it is the product of continuous functions and if $x \in D$, then $\overline{f}(x) = 0$ and if $a \in A$, then $\overline{f}(a) = h(a)\,g(a) = f(a)$ ■

9 Complete Spaces

Definition 258 *Let (X,d) be a metric space. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. $\{x_n\}_{n=1}^{\infty}$ is said to be a* ***Cauchy sequence*** *if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ whenever $n, m \geq N$*

Theorem 259 *If (X,d) is a metric space and $\{x_n\}_{n=1}^{\infty}$ is a sequence in X convergent to x. Then, $\{x_n\}_{n=1}^{\infty}$ is Cauchy*

Proof. Since $x_n \longrightarrow x$, $\exists N \in \mathbb{N}$ such that $d(x, x_n) < \epsilon/2$ so that if $n, m \geq N$, $d(x_n, x_m) \leq d(x, x_n) + d(x_m, x) < \epsilon$ ■

The converse is not generally true. If it is, then (X,d) is said to be **complete**.

Example 260 *$\mathbb{R}\backslash\{0\}$ is not complete: consider $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. We know that $\frac{1}{n} \longrightarrow 0$ so $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is Cauchy but is not convergent in $\mathbb{R}\backslash\{0\}$*

Example 261 *$(-1,1)$ is also not complete: the sequence $\left\{1 - \frac{1}{n}\right\}_{n=1}^{\infty}$ converges to 1 which is not in the set*

Example 262 *$\mathbb{Q}$ is also not complete. Since $\sqrt{2} \notin \mathbb{Q}$ but $\{1, 1.4, 1.414, 1.4142, ...\}$ is a Cauchy sequence since for $n > m$, $|x_n - x_m| < 10^{-m}$*

Lemma 263 *(X,d) is complete if and only if if every Cauchy sequence has a convergent subsequence*

Proof. $(\Longrightarrow)$ Trivial

$(\Longleftarrow)$ Let $\{x_n\}_{n=1}^{\infty}$ be Cauchy with a subsequence $\{x_{n_i}\}_{n=1}^{\infty}$ convergent to x. Let $\epsilon > 0$. Choose N so that if $m, n \geq N$, then $d(x_n, x_m) < \frac{\epsilon}{2}$. Choose $i \in \mathbb{N}$ so that $n_i \geq N$ and $d(x_{n_i}, x) < \epsilon/2$. So, $d(x_n, x) \leq d(x_n, x_{n_i}) + d(x, x_{n_i}) < \epsilon$ for $n \geq N$. ■

Lemma 264 *If (X,d) is a complete metric space and $A \subset X$ is closed, then A is complete.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in A. Then, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X so that $\exists x$ such that $x_n \rightarrow x \in X$. Since A is closed and $\{x_n\}_{n=1}^{\infty} \subset A$, it follows that $x \in A$ ■

Theorem 265 *For $k \geq 1$, $\mathbb{R}^k$ is complete.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be Cauchy. Then, $\exists N \in \mathbb{N}$ such $d(x_n, x_m) \leq 1$ so the subsequence $\{x_n\}_{n=N}^{\infty} \subset \overline{N_1(x_N)}$ so there is a subsequence $x_{n_i} \to x \in \overline{N_1(x_N)}$ so (X, d) is complete. ∎

$\mathbb{R}^J$, for J uncountable, is not metrizable in the product topology. If J is countable, then $\mathbb{R}^{\mathbb{N}}$ is complete. To show this, we use the following lemma.

Lemma 266 *Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of topological spaces and let $X = \prod_{\alpha\in A} X_\alpha$. Then $\{x_n\}_{n=1}^{\infty} \to x \in X \iff \{\pi_\alpha(x_n)\}_{n=1}^{\infty} \to \pi_\alpha(x)$ for each α*

Proof. $(\implies)$ $\{x_n\}_{n=1}^{\infty} \to x \implies \pi_\alpha : X \longrightarrow X_\alpha$ is continuous and $\pi_\alpha(x_n) \to \pi_\alpha(x)$

$(\impliedby)$ Let $x \in X$ and let $U \ni x$ be a basis neighborhood. So, $U = \prod_{\alpha\in A} U_\alpha$ where $U_\alpha = X_\alpha$ for all $\alpha \in A\backslash F$ where F is some finite subset of A. For each $\alpha \in F$, $\{\pi_\alpha(x_n)\}_{n=1}^{\infty} \to \pi_\alpha(x)$. Since $x \in U \implies \pi_\alpha(x) \in U_\alpha$ so $\exists N_\alpha \in \mathbb{N}$ so that if $n \geq N_\alpha$, then $\pi_\alpha(x_n) \in U_\alpha$. Let $N = \max_{\alpha\in F}\{N_\alpha\}$. If $n \geq N$, then for $\alpha \in F$, $\pi_\alpha(x_n) \in U_\alpha$ and if $\alpha \in A\backslash F$, then $\pi_\alpha(x_n) \in U_\alpha = X_\alpha$ and so $x_n \in U$. Since U is a basis element, $x_n \to x$. ∎

And now for a proof of the claim.

Proof. Let $\mathbb{R}^{\mathbb{N}}$ be equipped with the product topology. For $a, b \in \mathbb{R}$, define $\overline{d}(a, b) = \min(|a-b|, 1)$. This is a metric. Let $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$. Define $D(x, y) = \sup_{i\in\mathbb{N}} \left\{\frac{\overline{d}(x_i, y_i)}{i}\right\}$. Then, $\left(\mathbb{R}^{\mathbb{N}}, D\right)$ is complete. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Observe tht if $x, y \in \mathbb{R}^{\mathbb{N}}$, then $\overline{d}(\pi_i(x), \pi_i(y)) \leq iD(x, y)$. Since $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Let $\epsilon > 0$. $\exists N \in \mathbb{N}$ such that for $m, n \geq N$, $D(x_n, x_m) < \frac{\epsilon}{i} \implies \overline{d}(\pi_i(x), \pi_i(y)) \leq \epsilon$ so $\{\pi_i(x_n)\}_{n=1}^{\infty}$ is also Cauchy for each i. Hence there exists a limit point x_i of $\{\pi_i(x_n)\}_{n=1}^{\infty}$. If we let $x = (x_1, x_2, ...)$ and use the lemma above. ∎

9.1 Completion of Spaces

We can also complete a metric space. Intuitively, this is done by "adding" limit points so that every Cauchy sequence converges. Of course we can't include elements in the set on our own accord but what we can do is make the set "equal" to another complete subset of another space. This "equality" is not the true equality which we are wired to think of and is based on the definition of a type of isomorphism which follows this paragraph. This strange "equality" of sets makes the two sets indiscernable with respect to their structure and properties but the substance itself differs. For instance, as is known, the set $\mathbb{Q}$ is constructed from $\mathbb{Z}\times(\mathbb{Z}-\{0\})$. The removal of 0 ensures that zero is excluded in the denominator. The resulting set $\mathbb{Q}$ is not TRULY equal to an extension of the set of integers so that it is unfair to state that $\mathbb{Z} \subset \mathbb{Q}$, strictly speaking. $\mathbb{Z}$ happens to a single set whereas $\mathbb{Q}$ is the Cartesian product of both. What can actually be said is that $\mathbb{Z}\times\{1\} \subset \mathbb{Q}$. However, there exists an isomorphism between $\mathbb{Z}\times\{1\}$ and $\mathbb{Z}$, so that the sets are equal in their properties and structure but not substance. For all practical purposes, people usually don't beat about the bush and simply state $\mathbb{Z} \subset \mathbb{Q}$, which is safe to say because of the concept of isomorphism. To make the point relatable to completion, it is not true, strictly speaking, that $\bar{\mathbb{Q}} = \mathbb{R}$ because $\mathbb{Q}$ has holes in it and we have absolutely no authority to add points to complete the set of rational numbers but what we can do is make the set $\mathbb{Q}$ isomorphic to subset of complete set. Thus, while we're not really adding points, we're still making the set complete by accounting for the missing holes, which is why we have the colloquial "addition of points".

Here is the promised definition:

Definition 267 *A mapping T from a metric space (X,d) into $\left(\hat{X},\hat{d}\right)$ is said to be an **isometry** if T $\forall x,y \in X$, $\hat{d}(T(x),T(y)) = d(x,y)$. Two metric spaces are said to be **isometric** or **isomorphic** as metric spaces if there is a bijective isometry between them.*

If such a bijective mapping $T : X \to \hat{X}$ can be found, then X is said to be isometric with $\hat{X}$. This is the isomorphism for two metric spaces. Intuitively, an isometry preserves distances so nearby points in one space and equivalently near in another metric space. Remember, in metric spaces, one is concerned with distance between two points so that if two metric spaces have the same structure and properties, i.e. are isometric, then the distance between two points must be conserved and nothing else matters – not the names of the points, at least.

In order to complete any metric space, we can show that it is isomorphic to a dense subspace of a complete metric space and that this complete metric space must necessarily exist. Furthermore, this metric space is unique (up to isomorphism).

Theorem 268 *If (X,d) is a metric space then there is a complete metric space (X',d') such that $i : (X,d) \longrightarrow (X',d')$ is an isometric embedding.*

That is, for a metric space (X,d), there exists a complete metric space $\left(\hat{X},\hat{d}\right)$ which has a subspace W that is isometric with $\hat{X}$ such that $\bar{W} = \hat{X}$. Furthermore, this space is unique except for isometries.

Proof. First, we focus on the construction of $\left(\hat{X},\hat{d}\right)$. Let x_n and $\acute{x}_n$ be Cauchy sequences in X. We will call two Cauchy sequences equivalent if they have the same limit i.e.

$$\lim_{n\to\infty} d(x_n, \acute{x}_n) = 0$$

This will be written as $(x_n) \sim (\acute{x}_n)$. We can then gather all such equivalent sequences and form an equivalent class. Indeed, $(x_n) \sim (x_n)$ is trivial, so this relation is reflexive. Also, since the arguments of a metric function are symmetric, the relation $\sim$ is symmetric. Finally, if $(x_n) \sim (y_n)$ and $(y_n) \sim (z_n)$, we have

$$d(x_n, z_n) \leq d(x_n, y_n) + d(x_n, y_n)$$

Taking limits on both sides and using the fact that the metric function is always positive, we have

$$\lim_{n\to\infty} d(x_n, z_n) = 0$$

so that $(x_n) \sim (z_n)$, implying transitivity. Thus, we can have for ourselves an equivalence class $\hat{x} = \{\bar{x}_n\}$ of Cauchy sequences. We can collect all such equivalence classes $\hat{x}, \hat{y}, \ldots$ and form the set $\hat{X}$. For this set, we can have the metric function

$$\hat{d}(\hat{x}, \hat{y}) = \lim_{n\to\infty} d(x_n, y_n)$$

where $x_n \in \hat{x}$ and $y_n \in \hat{y}$. Note that this is not equal to zero since x_n and y_n are members of a different equivalence class. To show that this limit is well-defined or that this definition is sensible and not ambiguous with different results for the same choice of inputs, we will first show that this limit exists and then show that it is independent of the choice of representatives. First, we have

$$d(x_n, y_n) \le d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$$

$\Longrightarrow$

$$d(x_n, y_n) - d(x_m, y_m) \le d(x_n, x_m) + d(y_m, y_n)$$

Similarly,

$$d(x_m, y_m) \le d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m)$$

$\Longrightarrow$

$$d(x_m, y_m) - d(x_n, y_n) \le d(x_m, x_n) + d(y_n, y_m)$$

$\Longrightarrow$

$$-(d(x_m, x_n) + d(y_n, y_m)) \ge d(x_n, y_n) - d(x_m, y_m)$$

this is basically $b \ge a$ and $-b \ge a$ so that we have $|a| \le b$. Hence,

$$|d(x_n, y_n) - d(x_m, y_m)| \le d(x_n, x_m) + d(y_m, y_n)$$

Now, since x_n is Cauchy, we have $d(x_n, x_m) < \epsilon/2$ and similarly $d(y_m, y_n) < \epsilon/2$. This in turn implies that for $n, m > N$

$$|d(x_n, y_n) - d(x_m, y_m)| < \epsilon$$

so that

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{m\to\infty} d(x_m, y_m)$$

Hence, $\hat{d}(\hat{x}, \hat{y})$ is just as valid for any Cauchy sequence. Now, we prove that $\left(\hat{X}, \hat{d}\right)$ is a metric space. $\hat{d}(\hat{x}, \hat{y}) = 0 \iff \lim_{n\to\infty} d(x_n, y_n) = 0 \iff \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$ so that $(x_n) \sim (y_n)$, making them members of the same equivalence class. Since members of an equivalence class are either disjoint or the same, therefore $\hat{x} = \hat{y}$. Next, since $d(x_n, y_n) \ge 0$, we have $\hat{d}(\hat{x}, \hat{y}) \ge 0$. Furthermore, $d(x_n, y_n) = d(y_n, x_n)$ so that $\hat{d}(\hat{x}, \hat{y}) = \hat{d}(\hat{y}, \hat{x})$. Finally,

$$d(x_n, z_n) \le d(x_n, y_n) + d(y_n, z_n)$$

so that $\hat{d}$ obeys the triangle inequality.

We have just proved that for any metric space (X, d), we will have another metric space $(\hat{X}, \hat{d})$ by accounting for the limits of the Cauchy sequences, made possible by clumping all Cauchy sequences with common limits. Let $W \subset \hat{X}$ and let $T : X \longrightarrow W$ be a mapping such that $T(a) = \hat{a}$ where $\hat{a}$ is an equivalence class of constant Cauchy sequences. Here, W is a subclass of constant Cauchy sequences. Since if two sequences are both constant and converge to the same limit, then the two sequences are equal. Thus, every equivalence class of constant Cauchy sequences will be a singleton so that $\hat{b}$ will only contain the Cauchy sequence $(b, b, ...)$. We will now prove that this is an isometry.

First, notice that the mapping is clearly onto. This can be understood by recalling how we arrived at $\hat{X}$ and hence W. Next, for $T(b_1) = T(b_2)$, we have $\hat{b}_1 = \hat{b}_2$ so that the mapping is one-to-one. Hence T is bijective. Finally, T is an isometry since

$$\hat{d}\left(T\left(a\right), T\left(b\right)\right) = \hat{d}\left(\hat{a}, \hat{b}\right) = \lim_{n \to \infty} d\left(a_n, b_n\right) = d\left(a, b\right)$$

To show that this W is dense in $\hat{X}$. For that, we need to show that the limit points of W are in $\hat{X}$. That is, if $\hat{x} \in \hat{X}$, we should have $\hat{d}(\hat{x}, x) < \epsilon \ \forall \epsilon > 0$ contained in W for $x \in W$. For $\hat{x} \in X$ and $(x_n) \in \hat{x}$. Now, for any Cauchy sequence x_n the inequality

$$d\left(x_n, x_N\right) < \epsilon/2$$

will be valid $\forall \epsilon > 0$ whenever $n > N$. For the constant sequence

$$\left(x_N, x_N, ...\right) = \hat{x}_N \in W$$

we have

$$\hat{d}(\hat{x}, x) = \lim_{n \to \infty} d\left(x_n, x_N\right) \leq \epsilon/2 < \epsilon$$

so that every neighbourhood of $\hat{x}$ will contain a point of W.

To show that $\hat{X}$ is complete, let $(\hat{x}_n)$ be any Cauchy sequence in $\hat{X}$. Now, since W is dense in $\hat{X}$, every point $\hat{x}_n \in \hat{X}$ and $\forall \epsilon > 0$, we can find a point $\hat{z}_n \in W$ so that $\hat{d}(\hat{x}_n, \hat{z}_n) < \epsilon$. We can choose this $\epsilon = 1/n$ so that the sequence $(\hat{z}_n)$ becomes Cauchy. This can be observed as follows:

$$\begin{aligned} \hat{d}(\hat{z}_m, \hat{z}_n) &\leq \hat{d}(\hat{z}_m, \hat{x}_m) + \hat{d}(\hat{x}_m, \hat{x}_n) + \hat{d}(\hat{x}_n, \hat{z}_n) \\ &< 1/m + \hat{d}(\hat{x}_m, \hat{x}_n) + 1/n \end{aligned}$$

Since the element of W, $\hat{z}_n$, is Cauchy, $(z_m) = T^{-1}\left(\hat{z}_m\right)$ is also Cauchy in X. If (z_m) is contained in the class $\hat{x}$, then

$$\begin{aligned} \hat{d}(\hat{x}_n, \hat{x}) &\leq \hat{d}(\hat{x}_n, \hat{z}_n) + \hat{d}(\hat{z}_n, \hat{x}) \\ &< 1/n + \hat{d}(\hat{z}_n, \hat{x}) \\ &= 1/n + \lim_{m \to \infty} d\left(z_n, z_m\right) \end{aligned}$$

Since the sequence (z_m) is an element of the equivalence class of Cauchy sequences $\hat{x}$ and $\hat{z}_n$ is an equivalence class of Cauchy sequences and is contained in W, we have $(z_n, z_n, z_n, ...) \in \hat{z}_n$ and thus the inequality can be made as small as we like, implying that the limit of $\hat{x}_n$ is $\hat{x}$

If $\left(\tilde{X}, \tilde{d}\right)$ is another complete space with a subspace $\tilde{W}$ which is isometric with X such that $\tilde{W}$ is dense in $\tilde{X}$. Then, for any $\tilde{x}, \tilde{y} \in \tilde{X}$, we apply the same method as above to get

$$\left|\tilde{d}\left(\tilde{x}, \tilde{y}\right) - \tilde{d}\left(\tilde{x}_n, \tilde{y}_n\right)\right| \leq \tilde{d}\left(\tilde{x}, \tilde{x}_n\right) + \tilde{d}\left(\tilde{y}, \tilde{y}_n\right)$$

so that $\tilde{d}\left(\tilde{x}_n, \tilde{y}_n\right) \longrightarrow \tilde{d}\left(\tilde{x}, \tilde{y}\right)$, implying that $\hat{X}$ and $\tilde{X}$ are isometric. ■

10 Function Spaces

Now that we've seen what $\mathbb{R}^k$ and $\mathbb{R}^{\mathbb{N}}$ are, what can we say about $\mathbb{R}^A$, where A is any arbitrary set? In particular, if A is uncountable, then the product topology on $\mathbb{R}^A$ is not metrizable, a fact that we will take on faith. Thus, it makes no sense to talk about completeness for $\mathbb{R}^A$.

Let us define a different metric on $\mathbb{R}^A$ that does not give rise to the product topology and then show that $\mathbb{R}^A$ is complete. Recall, that $\mathbb{R}^A = \{f : A \longrightarrow \mathbb{R}\} := Hom_{\mathcal{S}et}(A, \mathbb{R})$. Let (Y, d) be a metric space. We need to put a metric on $Y^A = Hom_{\mathcal{S}et}(A, Y)$. It can be shown that $\overline{d}(x, y) = \min\{d(x, y), 1\}$ is a metric, bounded and metrically equivalent to d. Thus, $\left(Y, \overline{d}\right)$ is complete $\iff$ (Y, d) is complete. If $f, g \in Hom_{\mathcal{S}et}(A, Y) = Y^A$, then $\overline{\rho}(f, g) = \sup\limits_{a \in A} \overline{d}(f(a), g(a))$. Why not use $\sup\limits_{a \in A} d(f(a), g(a))$? Then the metric is not bounded so that the supremum might not exist. Note that $\overline{\rho}(f, g) \leq 1$. $\overline{\rho}$ is called the uniform metric.

Theorem 269 *$\left(Y^A, \overline{\rho}\right)$ is a complete metric space.*

Proof. We first show that $\overline{\rho}$ is a metric.

D1

Since $\overline{d}(f(a), g(a)) \geq 0$ for each $a \in A$, we must have $\sup\limits_{a \in A} \overline{d}(f(a), g(a)) \geq 0$. Thus, $\overline{\rho}(f, g) \geq 0$.

D2

$\overline{\rho}(f, f) = \sup\limits_{a \in A} \overline{d}(f(a), f(a)) = \sup\limits_{a \in A}\{0\} = 0$. Conversely, if $\overline{\rho}(f, g) = 0$, then $\sup\limits_{a \in A} \overline{d}(f(a), f(a)) = 0$. In particular, since $\overline{d}(f(a), g(a)) \geq 0$ for each $a \in A$, so that $\sup\limits_{a \in A} \overline{d}(f(a), g(a)) = 0 \implies \overline{d}(f(a), g(a)) = 0$. Again, this holds for each $a \in A$. $\implies f(a) = g(a)\ \forall a \in A \implies f = g$.

D3

$\overline{\rho}(f, g) = \sup\limits_{a \in A} \overline{d}(f(a), g(a)) = \sup\limits_{a \in A} \overline{d}(g(a), f(a)) = \overline{\rho}(g, f)$

D4
Since $\overline{d}(f(a), g(a)) \leq \overline{d}(f(a), h(a)) + \overline{d}(h(a), g(a))$ for each $a \in A$

$$\begin{aligned} \Longrightarrow \quad & \overline{d}(f(a), g(a)) \leq \overline{d}(f(a), h(a)) + \sup_{a\in A}\overline{d}(h(a), g(a)) \\ \leq \quad & \sup_{a\in A}\overline{d}(f(a), h(a)) + \sup_{a\in A}\overline{d}(h(a), g(a)) \end{aligned}$$

$$\Longrightarrow \sup_{a\in A}\overline{d}(f(a), g(a)) \leq \sup_{a\in A}\overline{d}(f(a), h(a)) + \sup_{a\in A}\overline{d}(h(a), g(a))$$

$$\Longrightarrow \overline{\rho}(f, g) \leq \overline{\rho}(f, h) + \overline{\rho}(h, g)$$

Let $\{f_n\}$ be a Cauchy sequence in Y^A. For $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that if $m, n \geq N$, $\overline{\rho}(f_n, f_m) < \epsilon$. So, for each $a \in A$, if $m, n \geq N$, then $\overline{d}(f_n(a), f_m(a)) \leq \overline{\rho}(f_n, f_m) < \epsilon$ so $\{f_n(a)\}$ is a Cauchy sequence for each $a \in A$. Since Y is complete, $\exists y_a \in Y$ such that $f_n(a) \to y_a$. Define $f \in Hom_{\mathcal{S}et}(A, Y)$ by $f(a) = y_a$. We claim that $f_n \to f$ uniformly.

First, we show ordinary convergence. Let $\epsilon > 0$. We can pick $N \in \mathbb{N}$ so that if $m, n \geq N$, then $\overline{\rho}(f_n, f_m) < \frac{\epsilon}{2}$ and $\overline{d}(f_n(a), f_m(a)) < \frac{\epsilon}{2}$. Since $f_n(a) \to f(a) = y_a$, it follows that for $n \geq N$, $\overline{d}(f_n(a), f(a)) \leq \frac{\epsilon}{2} < \epsilon$ so for each a and each $n \geq N$, $\overline{d}(f_n(a), f(a)) \leq \frac{\epsilon}{2}$ so that $\overline{\rho}(f_n, f) \leq \frac{\epsilon}{2} < \epsilon$ and so $f_n \to f$.

The convergence is uniform because for each $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that for $n \geq N$, $\overline{\rho}(f_n, f) < \epsilon \implies \overline{d}(f_n(a), f(a)) < \epsilon$ for all $a \in A$ if $n \geq N$. ∎

We can get a richer situation if we endow A with a topology. If this is the case, then we can define $C(A, Y) \subset Hom_{\mathcal{S}et}(A, Y)$, the set of continuous functions from A to Y. We can use either metric on Y: insert last homework problem here.

Theorem 270 *$(C(A, Y), \overline{\rho})$ is a complete metric space.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $C(A, Y)$. Then, the sequence in Cauchy in $Hom_{\mathcal{S}et}(A, Y)$ and so $\exists f \in Hom_{\mathcal{S}et}(A, Y)$ such that $f_n \to f$ uniformly. f is continuous by a previous result (uniform limit theorem). ∎

This also shows that $C(A, Y)$ is closed in $Hom_{\mathcal{S}et}(A, Y)$!

There is another nice space, $B(A, Y) \subset Y^A$, the space of all bounded functions. Recall that $f : A \longrightarrow Y$ is bounded if $\operatorname{diam} f(A) < \infty$. This admits a nicer metric: $\rho(f, g) = \sup_{a\in A} d(f(a), g(a))$. It can be shown that $\overline{\rho}(f, g) = \min(1, \rho(f, g))$ so that $(B(A, Y), \rho)$ is also complete. Observe that if A is compact, then $C(A, Y) \subset B(A, Y)$.

In the preceeding section, we proved that of (X, d) is a metric space then there is a complete metric space (X', d') such that $i : (X, d) \longrightarrow (X', d')$ is an isometric immersion. If (X, d) is a metric space and $\Phi : (X, d) \longrightarrow (X', d')$ is an isometric embedding into a complete metric space, then $\overline{\Phi(X)} \subset X'$ is called a metric completion of X. The proof in the preceeding section employs an equivalence class of Cauchy sequences. This turns out to be a model for $\mathbb{R}$. Let us do something different and furnish a proof of the same theorem.

Proof. Let $b \in X$. Given $a \in X$, define $\varphi_a : X \longrightarrow \mathbb{R}$ by $\varphi_a(b) = d(x,a) - d(x,b)$.

We first prove the $|d(x,a) - d(x,b)| \leq d(a,b)$, the reverse triangle inequality.

Since $d(x,a) \leq d(x,b) + d(b,a)$, and $d(x,b) \leq d(x,a) + d(a,b) \implies$

$d(x,a) - d(x,b) \leq d(a,b)$ and $d(x,b) - d(x,a) \leq d(a,b)$ respectively.

Now, for every $a \in X$, $\varphi_a \in B(X,\mathbb{R})$ since $|\varphi_a(x)| \leq |d(x,a) - d(x,b)| \leq d(a,b)$ so that $|\varphi_a(x)|$ is bounded, independent of x.

We've seen that $B(X,\mathbb{R})$ is complete. Define $\Phi : X \longrightarrow B(X,\mathbb{R})$ by $\Phi(a) = \varphi_a$. We need to show that Φ is an isometric embedding. To show this, let $u, v \in X$. Then, $\rho(\Phi(u), \Phi(v)) = \rho(\varphi_u, \varphi_v) = \sup\limits_{a\in X} d(\varphi_u(a), \varphi_v(a)) = \sup\limits_{a\in X} |\varphi_u(a) - \varphi_v(a)|$

$= \sup\limits_{a\in X} |d(a,u) - d(u,v) - d(a,v) + (a,v)| = \sup\limits_{a\in X} |d(a,u) - d(a,v)| \leq d(u,v)$

so $\rho(\Phi(u), \Phi(v)) \leq d(u,v)$. Conversely, if $a = u$, then $|d(a,u) - d(u,v)| = d(u,v)$ so $d(u,v) \leq \sup\limits_{a\in X} |d(a,u) - d(a,v)| = \rho(\Phi(u), \Phi(v))$. Thus, we have that $\rho(\Phi(u), \Phi(v)) = d(u,v)$. ■

10.1 Characterization of Compactness in Metric Spaces

When is a metric space compact? Of course when every sequence has a convergent subsequence and if every open cover has a finite subcover and if every infinite set has a limit. If $X \subset \mathbb{R}^k$, then X is compact $\iff$ X is closed and bounded. More generally, X is compact if and only if X is complete and totally bounded.

Definition 271 *Let (X,d) be a metric space. (X,d) is* ***totally bounded*** *if for every $\epsilon > 0$, X can be covered by finitely many ϵ-balls.*

All compact metric spaces are totally bounded: the cover of ϵ-balls on each point has a finite subcover.

All totally bounded sets are bounded: cover X with $\frac{1}{2}$-balls $\left\{N_{\frac{1}{2}}(x_1), ..., N_{\frac{1}{2}}(x_n)\right\}$. If $a, b \in X$, then $\exists i, j \in \{1, ..., n\}$ so that $a \in N_{\frac{1}{2}}(x_i)$ and $b \in N_{\frac{1}{2}}(x_j)$ and so $d(a,b) \leq 1 + d(x_i, x_j)$. Then, $diam(X) \leq 1 + \max d(x_i, x_j)$

The converse is not true: $\left(\mathbb{R}, \overline{d}\right)$ is bounded because we can cover $\mathbb{R}$ with a single ball but not totally bounded since, for example, we cannot cover $\mathbb{R}$ using $\left\{N_{\frac{1}{2}}(x_1), ..., N_{\frac{1}{2}}(x_n)\right\}$.

We will characterize compactness in a variety of function spaces $C(X,Y)$ with (Y,d) being a complete metric space.

The following is a generalization of the Bolzano-Weierstrass Theorem.

Theorem 272 *Let (X,d) is a metric space and $A \subset X$ be a subspace. Then A is compact if and only if A is complete and totally bounded.*

Proof. ($\Longrightarrow$) Let $A \subset X$ be compact and let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in A. Since A is compact, it is sequentially compact so that $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence in A. Thus, A is complete. Since every compact space is totally bounded, we are done.

($\Longleftarrow$) Let $A \subset X$ be complete and totally bounded. Pick a sequence $\{x_n\}_{n=1}^{\infty}$ in A. We will show that A is sequentially compact by (a) using totally boundedness to construct a Cauchy sequence and (b) apply completeness to see that it converges.

We can cover A with finitely may 1-balls so there is a 1-ball, B_1, which contains infinitely many terms of our sequence. Let $J_1 = \{n \in \mathbb{N} : x_n \in B_1\}$. By construction, $|J_1| = \infty$. We can cover A with finitely many $1/2$-balls so there is a $1/2$-ball B_2 such that $J_2 = \{n \in J_1 : x_n \in B_2\}$ is infinite. Inductively, we can find $1/k$-ball, B_k, such that $J_k = \{n \in J_{k-1} : x_n \in B_k\}$ is infinite and so we have a collection $J_1 \supset J_2 \supset ....$ Let $n_1 \in J_1$. Since J_2 is infinite, pick $n_2 \in J_2$ so that $n_2 > n_1$. Inductively, pick $n_k \in J_k$ such that $n_k > n_{k-1}$. We now need to show that $\{x_{n_k}\}$ is Cauchy. Let $\epsilon > 0$. Pick $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\epsilon}{2}$. If $k, l \geq N$, then x_k and x_l live in a $\frac{1}{N}$-ball and so $d(x_k, x_l) < \frac{2}{N} < \epsilon$. ■

Definition 273 *Let (X, τ) be a topological space and let (Y, d) be a metric space. Let $F \subset C(X, Y)$. F is called **equicontinuous at** x if for every $\epsilon > 0$, there is a neighborhood $U \ni x$ so that if $y \in U$ and $f \in F$, $d(f(x), f(y)) < \epsilon$. F is equicontinuous if it equicontinuous at every $x \in X$.*

Example 274 *For a finite collection $\{f_1, ..., f_n\}$ of continuous functions, this set is always equicontinuous because for each $a \in X$ and each $1 \leq i \leq n$ and for each $\epsilon > 0$, there exists a neighborhood U_i of a such that if $x \in U_i$, then $d(f_i(x), f_i(a)) < \epsilon$. Letting $U = \bigcap_{n=1}^{n} U_i$ satisfies the definition: since $x \in U_i$ for each i, then $x \in U$.*

On the other hand, consider $F = \{f_n \in C(\mathbb{R}, \mathbb{R}) : n \in \mathbb{N}\}$ by $f_n(x) = x^n$. Let $a = 1$ and let $\delta > 0$, $y = 1 + \frac{\delta}{2} \in (1 - \delta, 1 + \delta)$. Then $f_n(y) = \left(1 + \frac{\delta}{2}\right)^n \to \infty$ and so $d(f_n(a), f_n(y)) \to \infty$ so there is no such neighborhood of 1.

Theorem 275 *Let (X, τ) be a topological space and let (Y, d) be a metric space. If $F \subset C(X, Y)$ is totally bounded with uniform metric, then F is equicontinuous.*

Proof. Suppose that F is totally bounded and let $\epsilon > 0$ and let $a \in X$ and let $\delta = \min\left(\frac{\epsilon}{3}, \frac{1}{3}\right)$. Cover F by finitely many δ-balls $\{N_\delta(f_1), ..., N_\delta(f_n)\}$ where δ is such that each f_i is continuous at a. That is, we can pick a neighborhood U of a so that if $x \in U$, then $d(f_i(x), f_i(a)) < \delta$. Let $f \in F$. Then $\exists i$ so that $\rho(f, f_i) < \delta$ and so $\overline{d}(f(a), f_i(a)) < \delta$ but $\delta < \frac{1}{3} < 1$ and $d(f_i(y), f(y)) < \delta$ for every $y \in X$. Let $x \in U$. Then, $d(f(x), f(a)) \leq d(f(x), f_i(x)) + d(f_i(x), f_i(a)) + d(f_i(a), f(a)) < \delta + \delta + \delta \leq \epsilon$. ■

The converse is not generally true unless both X, Y are compact.

Theorem 276 *Let (X,τ) be a compact topological space and let (Y,d) be a compact and complete metric space. If $F \subset C(X,Y)$ is equicontinuous, then it is totally bounded with either uniform metric or the* sup *metric.*

Proof. Since X is compact, $C(X,Y) \subset B(X,Y)$ and F is totally bounded with respect to the sup metric, ρ, if and only if it is totally bounded with respect to $\overline{\rho}$ since small balls are the same in both metrics and so, we will only work with ρ. For each $\epsilon > 0$, if we let $\delta = \frac{\epsilon}{4}$ and for any point $a \in X$, there is a neighborhood U_a of a such that if $x \in U_a$ and $f \in F$, then $d(f(x), f(a)) < \delta$. Since X is compact, we can cover X with $\mathcal{U} = \{U_1, ..., U_n\}$ where U_i is a neighborhood of a_i. Next, since Y is compact, we can cover Y with finitely many sets $\mathcal{V} = \{V_1, ..., V_m\}$ of diameter$< \delta$. Let $J = \{\alpha : \{1, ..., n\} \longrightarrow \{1, ..., m\}\}$. Observe that $|J| < \infty$, and that for each $f \in F$, $f(a_i) \in V_j$ for some $1 \leq j \leq n$. For each $f \in F$, we can define a function $\alpha \in J$ by $\alpha(i) = j$. Let J' be the collection of all such functions $\left\{\alpha \in J : \exists f \in F : f(a_i) \in V_{\alpha(i)}, 1 \leq i \leq n\right\}$. Then, $|J'| < \infty$. For each $\alpha \in J'$, pick $f_\alpha \in F$ so that $f_\alpha(a_i) \in V_{\alpha(i)}$. We claim that $\{N_\epsilon(f_\alpha) : \alpha \in J'\}$ is a cover of F. Let $f \in F$. Pick a function in J' that is induced by f. Pick α so that $f(a_i) \in V_{\alpha(i)}$. We now claim that $\rho(f, f_\alpha) < \epsilon$ for $f \in N_\epsilon(f_\alpha)$. Let $x \in X$. Then, $\exists i \in \{1, ..., n\}$ so that $x \in U_i$. Then, for each $x \in X$, $d(f(x), f_\alpha(x)) \leq d(f(x), f(a_i)) + d(f(a_i), f_\alpha(a_i)) + d(f_\alpha(a_i), f_\alpha(x)) < 3\delta = \frac{3}{4}\epsilon$ and $\rho(f, f_\alpha) \leq \frac{3}{4}\epsilon < \epsilon$. ■

For a compact topological space X, when is $F \subset C(X, \mathbb{R}^n)$ compact?

Definition 277 *Let (X,τ) be a topological space. For $F \subset C(X,\mathbb{R}^n)$, F is **point-wise bounded** if for every $a \in X$, $F_a = \{f(a) : f \in F\}$ is bounded.*

Theorem 278 (Ascoli's Theorem) *Let (X,τ) be a compact topological space and $F \subset C(X,\mathbb{R}^n)$. Then, $\overline{F}$ is compact if and only if F is equicontinuous and point-wise bounded.*

Proof. ($\Longrightarrow$) Since X is compact, continuous functions from X to $\mathbb{R}^n$ are all bounded, so we can use the sup metric. Since $\overline{F}$ is compact, it is also totally bounded and hence $\overline{F}$ is equicontinuous and so F is also equicontinuous. Since $\overline{F}$ is totally bounded, it is also bounded and $\overline{F}$ bounded, by definition, implies that $\overline{F}_a$ is bounded for each $a \in X$ so that F_a is also bounded.

($\Longleftarrow$) Suppose that F is equicontinuous and point-wise bounded. Then, $\overline{F}$ is also equicontinuous and is point-wise bounded. To see this, first we prove that $\overline{F}$ is equicontinuous. Let $a \in X$ and let $\epsilon > 0$ be given. By equicontinuity of F, we can choose a neighborhood U containing a such that $d(f(x), f(a)) < \frac{\epsilon}{3}$ for all $x \in U$ and any $f \in F$. Choose $g \in \overline{F}$. Then, we can choose $f \in F$ such that $\rho(f,g) < \frac{\epsilon}{3}$ and for each $x \in U$, $d(g(x), g(a)) \leq d(g(x), f(x)) + d(f(x), f(a)) + d(f(a), g(a)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$.

To show that $\overline{F}$ is point-wise bounded, choose M so that $diam(F_a) < M$. Pick $g, g' \in \overline{F}$ and $f, f' \in F$ such that $\rho(f,g) < 1$ and $\rho(f', g') < 1$. Then, $d(g(x), g'(a)) \leq d(g(x), f(a)) + d(f(a), f'(a)) + d(f'(a), g'(a)) < 1 + M + 1 = M + 2$. Hence $diam\left(\overline{F}\right) < M + 2$.

Next, if we knew that the codomain is compact, then we would be done. Let us tweak the set-up. We need to show that there is a compact subset $Y \subset \mathbb{R}^n$ such that $\bigcup_{g\in\overline{F}} g(x) \subset Y$. In other words, $\overline{F} \subset C(X,Y)$. Since $\overline{F}$ is equicontinuous for each $a \in X$, we can pick U_a containing a so that $d(f(x), f(a)) < 1$ for all $x \in U_a$ and $f \in \overline{F}$. Since X is compact, we can cover X with finitely many $\{U_{a_1}, ..., U_{a_n}\}$. Since $\overline{F}$ is point-wise bounded, we can pick a constant K so that $\bigcup_{i=1}^{n} \overline{F}_{a_i} \subset N_K(0)$. Now, we can observe that, for $x \in X$, there has to be $i \in \{1, ..., n\}$ such that $x \in U_{a_i}$. Now, for $g \in \overline{F}$, we have $d(0, g(x)) \leq d(0, g(a_i)) + d(g(a_i), g(x)) < K+1$ so that $\bigcup_{g\in\overline{F}} g(x) \subset N_{K+1}(0)$. Letting $Y = \overline{N_{K+1}(0)}$ shows us that $\overline{F} \subset C(X,Y)$. Since $\overline{F}$ is a closed subset of the complete metric space $(C(X,Y), \rho)$, we see that $(\overline{F}, \rho)$ is a complete metric space. Since $\overline{F}$ is equicontinous, $\overline{F}$ is totally bounded. Since $(\overline{F}, \rho)$ is complete and totally bounded, then $(\overline{F}, \rho)$ is compact. ■

Note that the only property of $\mathbb{R}^n$ used in the proof is that $\overline{N_{K+1}(0)}$ is compact. That is, a set in which balls have compact closure. We can now move forward with a few generalizations of Ascoli's theorem. For that, a defintion.

Definition 279 *Let $F \subset C(X, \mathbb{R}^n)$. Then, F **vanishes at** ∞ if for every $\epsilon > 0$, there is a compact $C \subset X$ so that if $x \in X \backslash C$, then $|f(x)| = d(0, f(x)) < \epsilon$ for every $f \in F$. If $F = \{f\}$, then we say that f vanishes at ∞.*

We've seen that if X is locally compact and Hausdorff, then it has a unique one-point compactification in Y. How is this related to the previous definition? We can take each function $F \subset C(X, \mathbb{R}^n)$ and enlarge its domain. The goal is to take a function $f \in F \subset C(X, \mathbb{R}^n)$ which vanishes at infinity, and extend it to $\overline{f} \in C(Y, \mathbb{R}^n)$ such that $\overline{f}(x) = f(x)$ if $x \in X$ and 0 otherwise. This tells us the following:

1. If f vanishes at ∞, then $\overline{f} \in C(Y, \mathbb{R}^n)$
2. If F vanishes uniformly at ∞, then $G = \left\{\overline{f} \in C(Y, \mathbb{R}^n) : f \in F\right\}$ is equicontinuous at ∞

This suggests the following:

Theorem 280 *Let (X, d) be a locally compact and Hausdorff space and let $F \subset C(X, \mathbb{R}^n)$. Then, $\overline{F}$ is compact if and only if F is equicontinuous and point-wise bounded.*

Proof. If $\overline{F}$ is compact, then $\overline{F}$ is equicontinuous and point-wise bounded: cover $\overline{F}$ with $\left\{N_{\frac{\epsilon}{2}}(f_1), ..., N_{\frac{\epsilon}{2}}(f_n)\right\}$. For each $f \in F$, there is $i \in \{1, ..., n\}$ such that $\overline{\rho}(f, f_i) < \frac{\epsilon}{2}$. For each $i \in \{1, ..., n\}$, there is a compact C_i so that if $x \in X \backslash C_i$, then $|f(x_i)| < \epsilon/2$. Let $C = \bigcup_{i=1}^{n} C_i$, then if $x \in X \backslash C$, then $|f(x)| = |f(x) - f_i(x) + f_i(x)| \leq |f(x) - f_i(x)| + |f_i(x)| < \epsilon$, so that F vanishes uniformly at ∞. Define $\phi : \overline{F} \longrightarrow C(Y, \mathbb{R}^n)$ by $\phi(f) = \overline{f}$. ϕ is an isometric embedding with closed image of ϕ. In this case, $\phi(\overline{F})$ is equicontinuous and point-wise bounded. Thus, $\overline{\phi(\overline{F})}$ is compact. But $\overline{\phi(\overline{F})} = \phi(\overline{F})$ and also $\overline{F}$ is also compact. Let $f, g \in \overline{F}$, then $\sup_{a \in Y} d(\overline{f}(a), \overline{g}(a)) = \sup_{a \in X} d(f(a), g(a))$ and so $\rho_X(f, g) = \sup_{a \in \overline{F}} d(f(a), g(a)) = \sup_{a \in \phi(\overline{F})} d(\overline{f}(a), \overline{g}(a)) = \rho_Y(\overline{f}, \overline{g})$.

Now, suppose that $\{f_n\}$ in $\overline{F}$ and $\{\overline{f_n}\} \longrightarrow f \in \overline{\phi(\overline{F})} \subset C(Y, \mathbb{R}^n)$. We need to show that $f = \overline{g}$ for some $g \in \overline{F}$. Observe that $f(\infty) = 0$ so that $g = f|_X$. Does g vanish at infinity? Let $\epsilon > 0$. We know that $f_n \to f$ uniformly and so there is $N \in \mathbb{N}$ such that if $n \geq N$, then $\left|\overline{f}_n(x) - f(x)\right| < \frac{\epsilon}{2}$ for any $x \in Y$. We also know that for each $n \in \mathbb{N}$, there is a compact $C \subset X$ so that if $x \in X \backslash C$, then $|f_n(x)| < \epsilon/2$ so that $|f(x)| = |f(x) - f_n(x) + f_n(x)| \leq |f(x) - f_n(x)| + |f_n(x)| < \epsilon/2 + \epsilon/2 = \epsilon$. Hence f vanishes at infinity. Thus, $g \in \overline{F}$ and so $\phi(\overline{F})$ is closed. ■

There is another topology on the space of functions, other than the uniform topology, which is better suited for applications, specifically Homotopy Theory.

Let (Y, d) be a metric space and let X be a topological space. We have already seen that there is one topology, called uniform topology, on $C(X, Y)$. We'll now impose a slightly weaker condition: instead of saying that the functions are uniformly continuous everywhere, we let $K \subset X$ be compact, $\epsilon > 0$ and we also let $f \in Y^X$. Consider the set $B_K(f, \epsilon) = \left\{ g \in Y^X : \sup_{x \in K} d(f(x), g(x)) < \epsilon \right\}$. Then, there is a map from $Y^X \longrightarrow Y^K$ given by restriction $B_K(f, \epsilon)$ is the preimage of a ball in $C(K, Y)$. This topology is called the **topology of compact convergence** (CC topology) or even the **topology of uniform convergence on compact sets**.

Lemma 281 *Let $\mathcal{B} = \left\{ B_K(f, \epsilon) : K \subset X,\ K \text{ compact}, \epsilon > 0, f \in Y^X \right\}$. Then, $\mathcal{B}$ is a basis for a topology.*

Proof. If $f \in Y^X$, then $f \in B_K(f, \epsilon)$ for any compact K and $\epsilon > 0$. Next, suppose that $h \in B_K(f, \epsilon) \cap B_{K'}(f', \epsilon')$.

For $g \in B_K(f, \epsilon)$ and $\delta = \epsilon - \sup_{x \in K} d(f(x), g(x))$, let $\phi \in B_K(g, \delta)$. $\sup_{x \in K} d(\phi(x), g(x)) < \delta$ and $\implies \sup_{x \in K} d(\phi(x), g(x)) + \sup_{x \in K} d(f(x), g(x)) < \epsilon$. Since $\sup_{x \in K} d(\phi(x), f(x)) \leq \sup_{x \in K} d(\phi(x), g(x)) + \sup_{x \in K} d(f(x), g(x))$, we must have $\sup_{x \in K} d(\phi(x), g(x)) < \epsilon$. Then, we have $B_K(g, \delta) \subset B_K(f, \epsilon)$.

Similarly, for $g \in B_{K'}(f, \epsilon')$ and for $\delta' = \epsilon' - \sup_{x \in K'} (f(x), g(x))$, then $B_{K'}(g, \delta') \subset B_{K'}(f, \epsilon')$. Now let $D = \min(\delta, \delta')$, then $B_{K \cup K'}(g, D) \subset B_K(g, \delta) \subset B_K(f, \epsilon)$. Similarly, $B_{K \cup K'}(g, D) \subset B_{K'}(f, \epsilon')$. Thus, $B_{K \cup K'}(g, D) \subset B_{K'}(f, \epsilon') \cap$

$B_K(f,\epsilon)$. Clearly, $h \in B_{K\cup K'}(g,D)$ since $\sup\limits_{x\in K\cup K'} d(h(x),g(x)) \leq \sup\limits_{x\in K\cup K'} d(h(x),g(x)) < D$. $\blacksquare$

Theorem 282 *If $\{f_n : X \longrightarrow Y\} \longrightarrow f$ for some $f \in Y^X$ in the CC topology, then for each compact $K \subset X$, $\Phi_K : Y^X \longrightarrow Y^K$ with $\Phi_K(f) = f|_K$, then $B_K(f,\epsilon) = \Phi_K^{-1}(B(\Phi_K(f),\epsilon))$*

Proof. Follows from definition $\blacksquare$

Definition 283 *A space X is called **compactly generated** if, $A \in 2^X$ is open whenever $A \cap K$ is open in K for each compact $K \subset X$.*

This definition works if open is replaced with closed.

Lemma 284 *Let (X,τ) be a either a locally compact topological space or first countable topological space. Then, (X,τ) is compactly generated.*

Proof. If X is locally compact and $A \subset X$, so that $A\cap K$ is open in K for every compact $K \subset X$, we need to show that $A \cap K$ is open in K. Let $x \in A$. Choose a neighborhood $U \ni x$ and pick a compact $K \subset X$ such that $x \in U \subset K$. By hypothesis, $A\cap C$ is open in C and $A\cap U = (A\cap C)\cap U$ is open in U but since U is open, $A \cap U$ is open in X and so $A \cap U$ is a neighborhood of X in A. So, x is an interior point.

Assume that X is first countable. Let B be a set such that $B \cap K$ is closed for every compact set $K \subset X$. We need to show that B is closed. Let $x \in \overline{B}$. Pick a sequence $\{x_n\}$ in B that converges to x. Let $K = \{x_n : n \in \mathbb{N}\} \cup \{x\}$. Then, K is compact and $B \cap K$ is closed, by hypothesis and is closed in K. Thus, $x \in B$ and so B is closed. $\blacksquare$

We know that the restriction of a continuous function is continuous. Is the converse true? That is, can you extend a continuous function? Here is one possibility:

Lemma 285 *Let (X,τ) be a compactly generated topological space and let $f \in Y^X$ such that $f|_K$ is continuous for any compact K in X. Then, f is continuous.*

Proof. Let $V \subset Y$ be open and let $U = f^{-1}(V)$. We need to show that U is open. For each compact $K \subset X$, we know that $f|_K \in Y^K$ is continuous and so $U_K = f|_K^{-1}(V)$ is open in K. But $U_K = U \cap K$. Thus, U is open in K for any compact K. Since X is compactly generated, therefore U is open in X and so, f is continuous. $\blacksquare$

Theorem 286 *Let (X,τ) be a compactly generated topological space and Y^X have the CC topology. Then, $C(X,Y)$ is closed in Y^X*

Proof. Let $f \in \overline{C(X,Y)}$. We need to show that f is continuous. It suffices to show that $f|_K$ is continuous for every compact $K \subset X$. Let K be such a set (compact in X!). For each $n \in \mathbb{N}$, $B_K\left(f,\frac{1}{n}\right)$ intersects $C(X,Y)$. Let $f_n \in$

$C(X,Y)$ be an element of the intersection. By construction, $\rho\left(f_n|_K, f|_K\right) < \frac{1}{n}$ and so, $f_n|_K \to f$ in the uniform topology on $C(X,Y)$. We know that uniform limits of continuous functions is continuous. Then, $f|_K$ is continuous. ■

In all of the preceeding discussion, we've used the fact that the codomain has a metric. Can we drop this assumption? Let (X,τ_X) and (Y,τ_Y) be topological spaces and let $K \subset X$ be compact and $U \subset Y$ be open. Define $S(K,U) = \left\{f \in Y^X : f(K) \subset U\right\}$. This gives a notion of how close two functions are, without relying on a metric. Such a collection does not form a basis, sadly. Good news is that it is a sub-basis for a topological called the compact open topology. Is this the same as the CC topology?

Theorem 287 *If (X,d) is a metric space, then compact open topology is the same as the CC-topology.*

Proof. Let $\mathcal{S} = \{S(K,U) : K \subset X\}$ be the sub-basis for the compact open topology and let $f \in S(K,U) \in \mathcal{S}$. Since f is continuous, $f(K)$ is compact subset of U. Let $\epsilon = d(f(c), Y \backslash U)$. Then, $B_K\left(f, \frac{\epsilon}{2}\right) \subset S(K,U)$ and so, the topology of compact convergence is finer than the compact open topology.

For the converse, consider a basis element $B_K(f,\epsilon)$. Each point $x \in X$ lies in a neighborhood V_x so that $f\left(\overline{V_x}\right)$ has diameter $< \epsilon$. Let $V_x = f^{-1}\left(N_{\frac{\epsilon}{4}}(f(x))\right)$. Then, $f\left(\overline{V_x}\right) \subset N_{\frac{\epsilon}{3}}(f(x)) = U_x$ has diameter $\frac{2\epsilon}{3}$. Since K is compact, cover K with $\{V_{x_1}, ..., V_{x_n}\}$. Let $K_i = K \cap \overline{V_{x_i}}$. Then, K_i is compact and $f(K_i) \subset U_{x_i}$. So, $f \in S(K_i, U_i)$. Finally, if $g \in S(K_i, U_i)$, then $\sup\limits_{x \in K_i} d(f(x), g(x)) < \epsilon$ and so $\bigcap\limits_{i=1}^{n} S(K_i, U_i) \subset B_K(f,\epsilon)$. Hence $\{S(K_i,U_i) : i \in \{1,...,n\}\}$ is a subbasis. ■

Here are some nice properties these guys have. There is a map $e : X \times C(X,Y) \longrightarrow Y$ with $e(x,f) = f(x)$. This is simply the evaluation function. If X is locally compact and Hausdorff, then e is continuous. This can be viewed as follows: since we have a continuous function $f : X \times Z \longrightarrow Y$, then we get $F : Z \longrightarrow C(X,Y)$ with $F(z) = f_z$ with $f_z(x) = f(x,z)$. Thus, $F(z)(x) = f(x,z)$. We can now state another result:

Theorem 288 *If $f : X \times Z \longrightarrow Y$ is continuous, then $F : Z \longrightarrow C(X,Y)$ is continuous. If X is locally compact and Hausdorff, then the converse is true, as well.*

Corollary 289 *Let (X,τ_X), (Y,τ_Y) be topological spaces and let $(C(X,Y),\tau)$ be the topology on continuous functions from X to Y. Define the evaluation map $e : X \times (C(X,Y),\tau) \longrightarrow Y$ by $e(x,f) = f(x)$. Show that, for any topology τ, τ is finer than the compact open topology.*

Proof. We know that, for topological spaces (X,τ_X), (Y,τ_Y), (Z,τ_Z), then $f : X \times Z \longrightarrow Y$ is continuous tells us that $F : Z \longrightarrow (C(X,Y),\tau_c)$ is continuous, where $F(z) = f_z$ for $f_z(x) = f(x,z)$ and τ_c is the compact open topology . Thus, for $e : (X,\tau_X) \times (C(X,Y),\tau) \longrightarrow (Y,\tau_Y)$ to be continuous, we must have $F : (C(X,Y),\tau) \longrightarrow (C(X,Y),\tau_c)$ is continuous, defined by

$F(f) = e_f$ where and $e_f(x) = e(x, f) = f(x)$ for each x. That is, $e_f = f$ so that F is the identity map. Since F is identity map and continuous, for each open set in τ_c, we must have an open set in τ. Thus, $\tau_c \subset \tau$. ■

Example 290 *A homotopy is a continuous function* $h : [0,1] \times [0,1] \longrightarrow Y$ *with the following properties:* $h(x, 0) = \gamma_0(x)$*,* $h(x, 1) = \gamma_1(x)$ *and* $h(0, t) = p$ *and* $h(1, t) = q$*, where* p, q *are end points of paths* γ_0 *and* γ_1*. Then, we can have* $H : [0,1] \longrightarrow C([0,1], Y)$ *with* $H(t) = \gamma_t$ *defined by* $\gamma_t(x) = H(x, t)$