Preface

Notes for MTG 5326.0001 Topology II taught at FSU in Spring 2019 by Dr. Sam Ballas.

In these notes, the notation $A \subset X$ means that A is a subset of X, which may also refer to being an improper subset and 2^X refers to the powerset of X, which we assume to always exist for any X. $\mathbb{C}$ and $\mathbb{R}$, respectively, refer to the set of complex, respectively real, numbers.

Syllabus

Text:	Algebraic Topology, Allen Hatcher
Eligibility:	Graduate standing or permission of the instructor.
Course Content:	This course will cover basic concepts in point set topology. It will loosely follow Munkres book
Grading:	The grade distribution for this course is as follows:

Homework	40%
Midterms	30%
Final Exam	30%

The following numerical grade will guarantee at least the corresponding letter grade, although depending on the performance of the class the grade cutoffs may be lower:

A: 90-100; B: 80-89; C: 70-79; D: 60-69; F: 0-59.

Plus or minus grades may be assigned. A grade of I (incomplete) will not be given to avoid a grade of F or to give additional study time. Failure to process a course drop will result in a course grade of F.

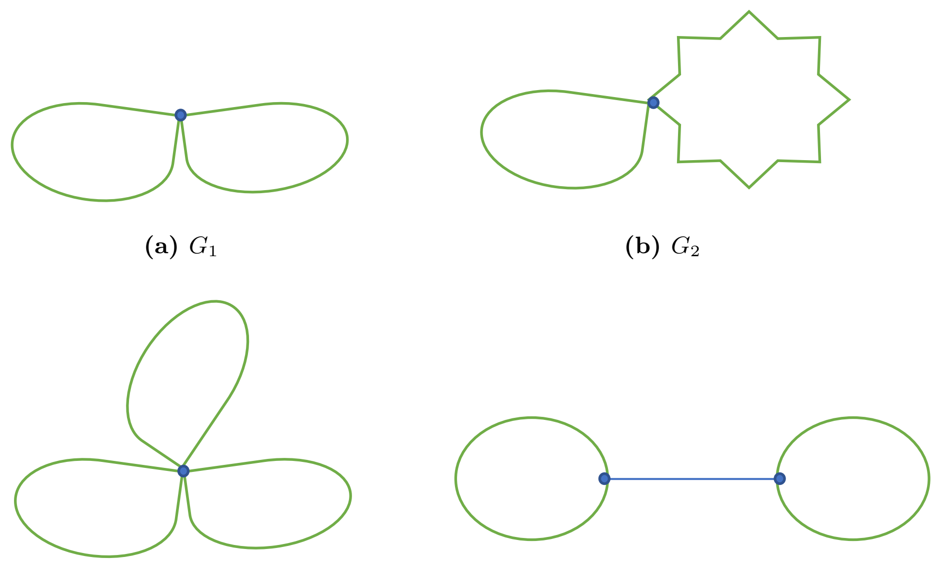
Exams: There will be a 2 midterm exams and a cumulative final. These exams will be taken during class. The exam schedule is as follows:

Exam 1:	Tuesday, February 19
Exam 2:	Tuesday, April 2
Final:	Tuesday, April 30 10a-12p

1 Introduction

Roughly, the main goal of Algebraic Geometry is to add algebraic objects to topological spaces such that the over-all structure is respected.

In this regard, what we should be looking for is a way to find invariant properties of topological objects. In case of graphs, which are topological objects, the Euler characteristic $\chi = v - e$ associates a number to each graph. Is this such an invariant? That is, do homeomorphic graphs have same number of vertices and edges and, therefore χ? By definition, of course! We refer the reader to Figure 1. G_1, the first graph and G_2, the second graph are both homeomorphic and they share the same Euler Characteristic. However, neither one of them is homeomorphic to G_3 and observe that they don't share the same χ. However, in the same figure, we have that two non-ismorphic graphs share the same χ. Thus, the converse is not true – two non-isomorphic graphs may have same χ. G_4 has the same Euler characteristic as G_1 and G_2. Thus, not all information is captured in χ. A similar situation happens in Algebraic Topology.



(a) G_1 **(b)** G_2

(c) G_3 **(d)** G_4

Figure 1: *Various graphs*

What information is the Euler characteristic losing? In other words, what is the equivalence relation between two graphs which share the same Euler characteristic? Such an equivalence is called **homotopy equivalence**. To define this, we first need to know what a homotopy is.

Definition 1 *A* ***homotopy*** *is a continuous map* $F : (X, \tau_X) \times [0, 1] \longrightarrow (Y, \tau_Y)$

From here on, all maps will assumed to be continuous.

Theorem 2 *F is a homotopy iff the homotopy is a family of maps* $f_t : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ *indexed by* $t \in [0,1]$

Definition 3 *If* (A, τ_A) *is a subspace of* (X, τ_X) *and there is a homotopy* $f_t : X \longrightarrow X$ *such that*

1. $f_0 = id_X$,
2. $f_1(X) \subset A$ *and*
3. $f_t = id_A$ *and* $\forall t \in [0,1]$

Then, f_t *is said to be a* ***deformation retraction*** *of* X *onto* A*. In this case,* A *is said to be a* ***deformation retract*** *of* X*.*

Example 4 *Let* $X = \{z \in \mathbb{C} : |z| < 2\}$ *and* $A = \{z \in \mathbb{C} : |z| \leqslant 1\}$*. Define*

$$f_t(x) = \begin{cases} x & \text{if } x \in A \\ \left(1 - t\left(\frac{1}{|x|}\right)\right) x & \text{if } x \notin A \end{cases}$$

Note that $f_0 = id_X$*. By definition,* $f_t = id_A$ *and* $\forall t \in [0,1]$*. Moreover, for* $t = 1$ *and* $|x| < 2$*, we have*

$$\frac{1}{|x|} > \frac{1}{2} \text{ so that } 1 - \frac{1}{|x|} < 1 - \frac{1}{2}$$

Thus, the scaling factor is at most $1/2$*, so that* $f_1(X) \subset A$*. We can thus imagine* f_t *squishing a disc to radius 2 to radius 1.*

What this example always suggests is that deformation retracts do not respect openness and closedness of sets and subsets.

Example 5 *What is a deformation retraction of* $\mathbb{R}^n \backslash \{0\}$ *onto the* $n-1$ *sphere* S^{n-1}*? Since we have* $\mathbb{R}^n$*, we can work with the Euclidean norm* $\|x\|^2 = x_1^2 + x_2^2 + ... + x_n^2$ *for* $x = (x_1, ..., x_n) \in \mathbb{R}^n$*. In fact, any norm would work since the topological spaces they generate are equivalent, as shown in Problem 1, Homework 1 last semester. We note that* $\mathbb{R}^n \backslash \{0\} \supset S^{n-1}$*.*

Define $f : \mathbb{R}^n \backslash \{0\} \longrightarrow \mathbb{R}^n \backslash \{0\}$ *by*

$$f(x) = \frac{x}{\|x\|}$$

This function is well-defined: ***Proof.*** *Let* $x = y$*. Then,* $\|x\| = \|y\|$ *and* $\frac{x}{\|x\|} = \frac{y}{\|y\|}$ *so that* $f(x) = f(y)$ ■

Now define $F : \mathbb{R}^n \backslash \{0\} \times [0,1] \longrightarrow \mathbb{R}^n \backslash \{0\}$ *by* $F(x,t) = tf(x) + (1-t)x$*.* F *is well-defined since multiplication and subtraction are well-defined in* $\mathbb{R}^n \backslash \{0\}$*. Denote* $F(x,t) = f_t(x)$*. Then,* $f_0(x) = (0)(f(x)) + (1-0)x = x$ *so that*

$f_0 = id_{\mathbb{R}^n \setminus \{0\}}$. *Furthermore,* $F(x,1) = f(x)$ *for* $x \in \mathbb{R}^n \setminus \{0\}$ *so that* $f(x) \in S^{n-1}$ *by definition of* $f(x)$ *and thus* $f_1(\mathbb{R}^n \setminus \{0\}) \subset S^{n-1}$. *Finally, for* $a \in S^{n-1}$,

$$f_t(a) = tf(a) + (1-t)a = t\frac{a}{\|a\|} + a - ta$$

Note that $\|a\| = 1$ *so that*

$$t\frac{a}{\|a\|} + a - ta = ta + a - ta = a$$

An even weaker notion of a deformation retract is what's called a retraction:

Definition 6 *If* (A, τ_A) *is a subspace of* (X, τ_X) *and* $r : X \longrightarrow A$ *is a map such that* $r|_A = id_A$, *then* r *is a* ***retraction***.

In lieu of $r|_A = id_A$, we can equivalently say that $r \circ i = id_A$, where $i : A \hookrightarrow X$ is the inclusion map.

If $f_t : (X, \tau_X) \longrightarrow (X, \tau_X)$ is a deformation retraction, then $f_1 : (X, \tau_X) \longrightarrow (A, \tau_A)$ is a retraction. The converse is not true, because of the notion of "time" in a deformation retraction. Consider any topological space (X, τ_X) and let $a \in X$ and now let $A = \{a\}$. Then, the (only!) function $r : (X, \tau_X) \longrightarrow (A, \tau_A)$ defined by $r(x) = a$ is a retraction but not a deformation: suppose that $r = f_t$ for some $f_t : X \longrightarrow A$ so we must have $f_0(x) = x$, $f_1(x) = a$ and $f_t(a) = a$ for each t. That is, we have a homotopy from id_X to the constant map r. Thus, the only way a retraction r is a deformation retract is when we have a homotopy from id_X to r.

Let (X, τ) be a topological space and $I = [0,1]$. Then, a continuous function $f : I \longrightarrow X$ is a **path**. For each $y \in X$, define a path $\gamma_y : [0,1] \longrightarrow X$ by $\gamma_y(t) = f_t(y)$. This is a path drawn by a fixed point in X. This path starts at y since $f_0(y) = y$ and ends at a since $f_1(y) = a$. In other words, we have just proved that if X deformation retracts to a point, then X is a path connected.

Definition 7 *A space* (X, τ_X) *is* ***contractible*** *if* X *deformation retracts onto a point.*

The point does not matter: as in the above proof, the choice $a \in X$ was arbitrary.

Problem 8 *Show that if* X *is contractible,* $A \subset X$ *and* $r : (X, \tau_X) \longrightarrow (A, \tau_A)$ *is a retraction, then* A *is also contractible.*

Solution 9 *Let* X *be contractible to a point* $s_0 \in X$. *Then, there exists a function* $F : X \times [0,1] \longrightarrow X$ *such that* $F(x,0) = x$, $F(x,1) \in \{s_0\}$ *and* $F(s_0, t) = s_0$ $\forall t \in [0,1]$ *Since* $r : X \longrightarrow A$ *is a retraction, then* $r|_A = id_A$. *That is,* $r(i(a)) = a$ *where* $i : A \hookrightarrow X$ *is the inclusion map. Define* $G : A \times [0,1] \longrightarrow A$ *by* $G(a,t) = F(r(i(a)), t)$. *This function is well-defined*

Proof. *Let* $(a_1, t_1) = (a_2, t_2)$. *Then,* $a_1 = a_2$ *and* $t_1 = t_2$ *so that and so* $r(i(a_1)) = r(i(a_2))$ *and* $F(r(i(a_1)), t_1) = F(r(i(a_2)), t_2)$ ■

Then, $G(a,0) = F(r(i(a)),0) = a$, $G(a,1) = F(r(i(a)),1) = s_0$ and $G(a,t) = F(r(i(a)),t) = s_0$ so that G is a deformation retraction to the point s_0.

Definition 10 *Let (X,τ_X) and (Y,τ_Y) be topological spaces, $A \subset X$ and $f_t : X \longrightarrow Y$ be a homotopy. Then, f_t is a* ***homotopy relative*** *to A (rel A) if $f_t|_A = f_0|_A = f_1|_A$ for all t.*

In this case, the notation is $f_0 \simeq_A f_1$ and that $f_t(a) = f_1(a) = f_0(a)$ for all $a \in A$. This is a useful notation in, for example, fixed end-points: let $X = [0,1]$ and Y be any topological space. Let $A = \{0,1\}$. This homotopy has fixed "end points". The usual definition of a homotopy is usually relative to $A = \varnothing$. Thus, if f_t is a homotopy such that $f_0 = g$ and $f_1 = h$, then it is not ambiguous to say that $g \simeq h$.

Proposition 11 *Let (X,τ_X) and (Y,τ_Y) be topological spaces and $f, f' : (X,\tau_X) \longrightarrow (Y,\tau_Y)$ and $g, g' : (Y,\tau_Y) \longrightarrow (Z,\tau_Z)$ be continuous maps. If $f \simeq f'$ and $g \simeq g'$, then $g \circ f \simeq g' \circ f'$.*

Proof. Let $F : (X,\tau_X) \times [0,1] \longrightarrow (Y,\tau_Y)$ be a homotopy between f, f' and $G : (Y,\tau_Y) \times [0,1] \longrightarrow (Z,\tau_Z)$ be a homotopy between g, g'. Define a map $H : X \times [0,1] \longrightarrow Z$ by $H = G(F(x,t),t)$. Clearly, H is continuous. Moreover, $H(x,0) = G(F(x,0),0) = G(f(x),0) = g(f(x))$ and $H(x,1) = G(F(x,1),1) = G(f'(x),1) = g'(f'(x))$. Thus, H is a homotopy between $g \circ f$ and $g' \circ f'$. ■

Thus, we can compose maps while respecting homotopies.

With these definitions in hand, we can now talk about homotopy equivalence, a weaker notion of homeomorphism.

Definition 12 *Let (X,τ_X) and (Y,τ_Y) be topological spaces. Then, (X,τ_X) and (Y,τ_Y) are* ***homotopy equivalent****, denoted as $(X,\tau_X) \simeq (Y,\tau_Y)$, if there are maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ so that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y.*

That is, $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. In this case, the spaces X and Y are also said to be homotopic and the context should shed away any cause of confusion.

If X and Y are homeomorphic and f is the homeomorphism between them, then $g = f^{-1}$ satisfies the above requirements. Thus, homeomorphism is a notion stronger than homotopy equivlence.

This is an equivalence relation.

Proof. Let $(X,\tau_X), (Y,\tau_Y)$ and (Z,τ_Z) be topological spaces. Consider a continuous function $id_X : (X,\tau_X) \longrightarrow (X,\tau_X)$. With the constant homotopy $f_t(x) = x$ for all t, we can say that $f_0 = id_X$ whereas $f_1 = id_X^{-1}$. Thus, $id_X \circ id_X^{-1} \simeq id_X$ so that $X \simeq X$.

If $(X,\tau_X) \simeq (Y,\tau_Y)$, then there are maps $f : X \longrightarrow Y$, $g : Y \longrightarrow X$ so that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. That is, there are maps $g : Y \longrightarrow X$ and $f : X \longrightarrow Y$ so that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. Thus, $(Y,\tau_Y) \simeq (X,\tau_X)$.

If $(X, \tau_X) \simeq (Y, \tau_Y)$ and $(Y, \tau_Y) \simeq (Z, \tau_Z)$, then we are guaranteed the existence of continuous maps $\exists$ continuous $f_1 : (X, \tau_X) \longrightarrow (Y, \tau_Y)$, $\exists$ continuous $g_1 : (Y, \tau_Y) \longrightarrow (X, \tau_X)$ such that $f_1 \circ g_1 \simeq id_Y$ and $g_1 \circ f_1 \simeq id_X$ and $\exists$ continuous $f_2 : (Y, \tau_Y) \longrightarrow (Z, \tau_Z)$, $\exists$ continuous $g_2 : (Z, \tau_Z) \longrightarrow (Y, \tau_Y)$ such that $f_2 \circ g_2 \simeq id_Z$ and $g_2 \circ f_2 \simeq id_Y$. Define $f : (X, \tau_X) \longrightarrow (Z, \tau_Z)$ by $f := f_2 \circ f_1$ and $g : (Z, \tau_Z) \longrightarrow (X, \tau_X)$ by $g = g_1 \circ g_2$. f and g are well-defined since f_1, f_2, g_1 and g_2 are well-defined and because the composition of functions is well-defined. Also, f and g are continuous since the compostion of functions is continuous. Finally, $f \circ g = (f_2 \circ f_1) \circ (g_1 \circ g_2)$

$= f_2 \circ (f_1 \circ g_1) \circ g_2$

$\simeq f_2 \circ id_Y \circ g_2$ by Proposition 11

$= f_2 \circ g_2$

$\simeq id_Z$

Similarly, $g \circ f = (g_1 \circ g_2) \circ (f_2 \circ f_1)$

$= g_1 \circ (g_2 \circ f_2) \circ f_1$

$\simeq g_1 \circ id_Y \circ f_1$

$= g_1 \circ f_1$

$\simeq id_X$

Thus, $(X, \tau_X) \simeq (Z, \tau_Z)$ ■

A **contractible space** is a space that is homotopy equivalent to a point: in one direction, we have the deformation retraction $r : X \longrightarrow \{a\}$ and on the other $i : \{a\} \hookrightarrow X$. Then, the maps are homotopic to the respective identities: by definiton, $r \circ i = id_{\{a\}}$ so that, in particular, $r \circ i \simeq id_{\{a\}}$. Conversely, $i \circ r \simeq id_X$ follows from the fact that r is itself a homotopy rel A from id_X to a retraction from X to A. This is because (2) and (3) in **Definition 3**.

We now go back to G_1 and G_4 in Figure 1. Smash the line in G_4 to a point by bringing the vertices together. Then, G_1 is a retract of G_4. Call this operation f. Conversely, think of the right circle of G_1. Take two points on the right circle and form a chord. Shrink the cord down by bringing its end points together. We now have three (distorted) circles joined together. and glue the arcs that form between these joined end points and the origin vertex of the graph. Call this complete operation g. These deformations make G_1 and G_4 homotopy equivalent because doing g first on G_1 and then applying f gives us G_1 back. That is, $f \circ g = id_{G_1}$. Similarly, $g \circ f = id_{G_4}$.

Theorem 13 *If G_1 and G_2 are two finite graphs, then G_1 and G_2 are homotopy equivalent iff their Euler characteristics are the same.*

In the example above, we collapsed an edge with distinct end points and get a homotopy equivalent graph with the same Euler characteristic. If we think about it, this is essentially what happens whenever we get a homotopy. If we keep on repeating this step, we will ultimately get a single vertex with k petals (something like G_3). Thus, every graph is homotopy equivalent to a rose G_k with k petals. That is, if G_k is a rose with k petals, then $\chi(G_k) = 1 - k$. Thus, we need to show that G_k is homotopy equivalent to G_l iff $k = l$. The proof of this fact will have to wait for now.

Note that the labelling in Figure 1 does not correspond to the definition of a graph G_k with k petals, except G_3 in Figure 1c.

2 CW Complexes

Before they're defined, let us look at examples: graphs! They are an example of 1-dimensional CW Complexes. Another class of examples are surfaces, built using a gluing construction. We will have a look at surfaces in more detail in later sections. For now, we simply focus on CW Complexes.

An n-cell is interior of an n-disc $D^n = \{x : |x| \leqslant 1\} \subset \mathbb{R}^n$. The boundary of D^n, ∂D^n, is $S^{n-1} = \{x : |x| = 1\} \subset \mathbb{R}^n$. That is, the boundary of an n-cell is an $n-1$ sphere. Thus, the empty set is a -1 sphere. A straight line is a 1-cell. A 0 sphere is just two dots, which are on the "end" of a straight line, a 1-cell. A 0 cell, D^0, is simply the set 0. A square, which is a graph, is a collection of cells. The vertices are 0-cells, the edges are 1-cells whereas the area is a 2-cell. CW Complexes are formed by n-dimensional cells. A square is thus a CW complex with four 0-cells, four 1-cells and one 2-cell.

What is the CW composition of a torus? A torus can be obtained by gluing a square: join opposite sides together to get a cylinder and then glue the edges of the cylinder to get a torus. This is a genus 1 surface. Informally, a genus is the number of handles of a surface.

This is not the only possible shape we can get with a square. We can glue each side of the square up to get a sphere, as well. To get yet another different surface, we can start with an octagon. These can be joined together to get a torus with two holes, a genus 2 surface. The labels of the octagon in Figure 2 correspond to sides which need to be glued. The inverse indicates reverse order of gluing sides (also indicated by arrows).

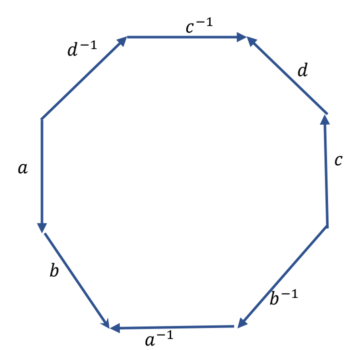


Figure 2: *Gluing construction for a 2-genus surface from an octogen*

The recipe for a CW Complex is as follows:

1. Start with a discrete set, X^0, called the 0-skeleton, which basically means

a set with the discrete topology. A lot of times, this will be finite but this is not necessary.

2. Assume we have defined the $n-1$ skeleton X^{n-1}.

3. Let $\{D^n_\alpha\}_{\alpha\in A}$ be a collection of n-cells. Define an attaching map $\varphi_\alpha : S^{n-1} = \partial D^n_\alpha \longrightarrow X^n$ where X^n is defined as

$$\frac{X^{n-1} \sqcup \{D^n_\alpha\}_{\alpha\in A}}{x \sim \varphi_\alpha(x)} \forall x \in \partial D^n_\alpha$$

In 3, the denominator implies that we are considering x the same as $\varphi_\alpha(x)$ and then obtaining the quotient topology. Thus, we can equivalently say that for each α and for each $x \in X$, the following diagram commutes

$$\begin{array}{ccc} \partial D^n_\alpha & \xrightarrow{\varphi_\alpha} & X^n \\ \big\downarrow i & & \big\downarrow \\ D^n_\alpha & \longrightarrow & \frac{X^{n-1} \sqcup D^n_\alpha}{x \sim \varphi_\alpha(x)} \end{array}$$

In the example Figure 3, $X^0 = \{1,2,3\}$ and the 1-cells are a,b,c,d. Call the indexing set $A = \{\alpha,\beta,\gamma,\delta\}$. Then, $D^1_\alpha = a$, $D^1_\beta = b$, $D^1_\gamma = c$ and $D^1_\delta = d$ such that $\partial D^1_\alpha = \{1,2\}$, $\partial D^1_\beta = \{2,3\}$, $\partial D^1_\gamma = \{1,3\}$ and $\partial D^1_\delta = \{3\}$. Then, $\varphi_\alpha : S^0 = \{1,2\} \longrightarrow X^0$ is as follows: $\varphi_\alpha(1) = 1$ and $\varphi_\alpha(2) = 2$ whereas $\varphi_\beta(2) = 2$ and $\varphi_\beta(3) = 3$ (is this right?)

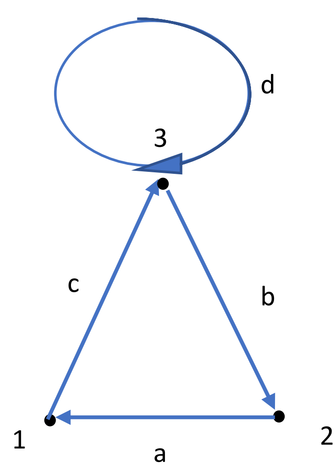


Figure 3: *Graph as a complex*

To get a 1-dimensional CW Complex S^1, consider the interval $[0,1] = a$. Then, a 1 cell a, a straight line and two 0-cells $0,1$ can be attached by a constant map $\varphi(1) = \varphi(0)$ to get a circle. We can get a "usual" sphere in $\mathbb{R}^3$, the 2-sphere S^2, as a 2-cell by first constructing the 1-dimensional CW Complex S^1 and then identifying each points of S^1 with a 2-cell D^2. Alternatively, we can just as well start with a 2-cell D^2 and a 0-cell a. And then identify all of ∂D^2 to the 0-cell a by a constant map. We thus get a S^2 with a on "top" of it. In general, to get the n-sphere, all we need is the 0-skeleton $X^0 = \{a\}$ and an n-cell. The process described above would then have empty i-skeletons $X^i = \varnothing$ for $1 \leqslant i \leqslant n-1$ whereas

$$X^n = \frac{D^n}{a \sim \varphi(\partial D^n)}$$

A projective space is also an example of a CW-Complex. The real projective n-space $\mathbb{R}_p^n$, or more popularly $\mathbb{R}P^n$, is the set of lines through the origin in $\mathbb{R}^{n+1}$. That is, $\mathbb{R}P^n$ is formed by taking the quotient of $\mathbb{R}^{n+1}\backslash\{0\}$ under the equivalence relation $x \sim \lambda x$ for all real numbers $\lambda \neq 0$. Since we can always rescale λ to unity, the quotient is then over $x \sim -x$, antipodal points. Thus, the real projective space is then the quotient of lines with end-points of each line identified.

$\mathbb{R}_p^0$ is the set of all lines in $\mathbb{R}^1$ which pass through the origin with $-\infty \sim \infty$. There is only one such line so that $\mathbb{R}P^1 = \{\mathbb{R}\}$. This is a singleton and thus a 0-cell. Therefore, $\mathbb{R}_p^0$ is a CW-Complex.

$\mathbb{R}_p^1$ is interesting. The set of all lines passing through the origin in a plane basically give us a single line itself because we can identify the slope of each line with a real number. With the end-points identified, we get a circle. Note that identifying antipodal points on (the boundary of) a circle makes no difference to the circle. That is, $S^1 \simeq \frac{S^1}{x\sim -x}$. $\mathbb{R}_p^2$ is a filled sphere with end-points identified. In general, $\mathbb{R}_p^n \simeq \frac{D^n}{x\sim -x}$ where $x \in \partial D^n$.

What are the subobjects of a CW-Complex? A subcomplex A of a CW Complex X is a closed subspace $A \subset X$ that is a union of cells. A CW subcomplex is itself a CW Complex. A pair (X, A) consisting of a CW Complex and a subcomplex A is called a **CW pair**. For example, a k-skeleton of a CW complex is a subcomplex. Thus, lower dimensional $\mathbb{R}_p$'s are subcomplexes of higher dimensional $\mathbb{R}_p$'s. A more interesting class of examples are spheres. Depending on the construction, you can find trivial or non-trivial CW subcomplexes.

3 Fundamental Group

Recall that for two finite graphs G_1 and G_2, we have still left **Theorem 13** hanging: G_1 and G_2 are homotopy equivalent iff their Euler characteristics are the same. That is, $\chi(G_1) = \chi(G_2)$. One direction of the proof was partially done (without many details?) To get the forward direction, we will apply what's called the fundamental group. This is a topological tool. The rough idea is that a fundamental group sees how many loops there are in a space. What the fundamental group does is takes a topological space as input and gives a group.

If two spaces are homotopic, then the groups are isomorphic. In cat speak, it is a functor from the category of topological spaces to the category of groups.

Two paths $f_0, f_1 : [0,1] \longrightarrow X$ have same end points if $f_0(0) = f_1(0) = x_0$ and $f_0(1) = f_1(1) = x_1$. Two paths are homotopic (rel x_0, x_1), denoted by $f_0 \simeq_{\{x_0,x_1\}} f_1$, if there is a homotopy from $F : [0,1] \times [0,1] \longrightarrow X$ such that (a) $F(0,t) = x_0$ (b) $F(1,t) = x_1$ (c) $F(s,0) = f_0(s)$ and (d) $F(s,1) = f_1(s)$ (see Figure 6)

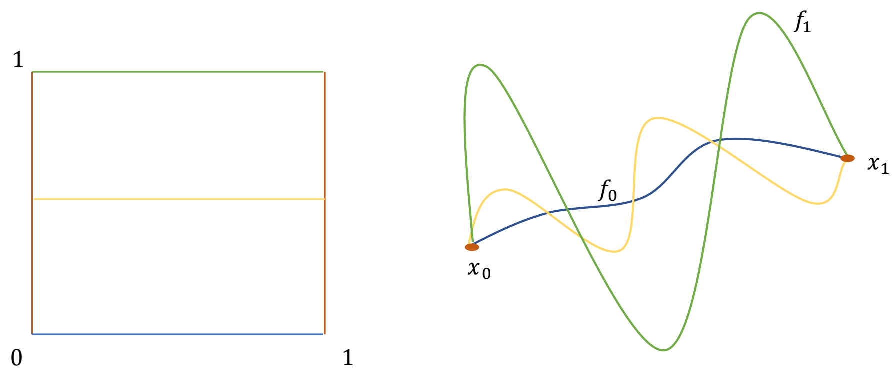


Figure 4: *Homotopy of paths*

For example, let $X = \mathbb{R}^n$ with the usual topology and let $x_0, x_1 \in \mathbb{R}^n$ be two points with paths f_0, f_1 between x_0 and x_1. Consider $F : [0,1] \times [0,1] \longrightarrow \mathbb{R}^n$, given by $F(s,t) = (1-t) f_0(s) + f_1(s)$. To decrypt this, consider again Figure 6. In this case, we have two paths with fixed end points. We start off with one path and wriggle it out to the other. This wriggling corresponds to adding two paths together by varying the weight (i.e., contribution) of each path.

This is so useful that it deserves its own theorem:

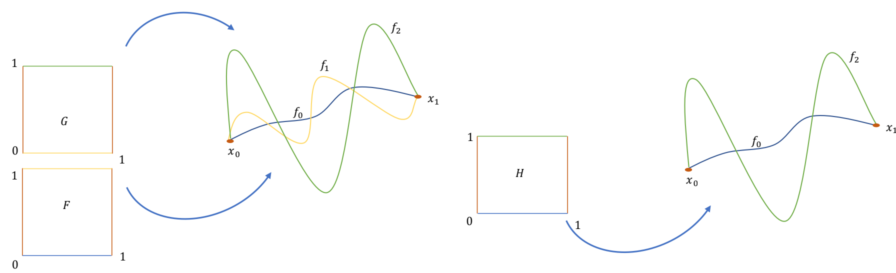
Theorem 14 *If $X \subset \mathbb{R}^n$ is a convex subset and $x_0, x_1 \in X$ and f_0, f_1 are paths in X between x_0 and x_1, then $f_0 \simeq_{\{x_0,x_1\}} f_1$ inside X.*

Proof. Since X is convex, for a fixed s, $F(s,t) = (1-t) f_0(s) + f_1(s)$ is always in X. Note that $F(0,t)$ is a straight line between $f_0(0)$ and $f_1(0)$ for each s. That is, a straight line between x_0 and x_0 if $s = 0$ and a straight line between x_1 and x_1 if $s = 1$. $F(s,t)$ is therefore a path for each t in X. Note that $F(s,t)$ is a homotopy between f_0 and f_1. ■

Thus, convex spaces do not have a very interesting homotopy structure.

The notation, $f_0 \simeq_{\{x_0,x_1\}} f_1$, suggests that being homotopic is an equivalence relation. This is indeed true. In what follows, the set $\{x_0, x_1\}$ in the subscript of $\simeq$ will be dropped.

Lemma 15 *Let X be a topological space and let x_0, x_1 be points in X. Then, the relationship of being homotopic is an equivalence relation.*



(a) *Hypothesis*

(b) *Conclusion*

Figure 5: *Transitive property of homotopy equivalence*

Proof. Reflexive: let f_0 be a path from x_0 to x_1. The homotopy $F(s,t) = f_0(s)$ is constant in t. Thus, $f_0 \simeq f_0$. Note that $F(0,t) = f_0(0) = x_0$, $F(1,t) = f_0(1) = x_1$ and $F(s,0) = f_0(s) = F(s,1)$.

Symmetry: if f_0 and f_1 be a paths from x_0 to x_1 and $f_0 \simeq f_1$. Then, $\exists F : [0,1] \times [0,1] \longrightarrow X$ such that $F(s,0) = f_0(s)$, $F(s,1) = f_1(s)$, $F(0,t) = x_0$ and $F(1,t) = x_1$. Define $H : [0,1] \times [0,1] \longrightarrow X$ given by $H(s,t) = F(s, 1-t)$. Then, $H(s,0) = f_1(s)$, $H(s,1) = f_0(s)$, $H(0,t) = x_1$ and $H(1,t) = x_0$ so that $f_1 \simeq f_0$.

Transitive: suppose that $f_0 \simeq f_1$ with end points x_0 and x_1 and $f_1 \simeq f_2$ with end points x_0 and x_1. Then, $\exists$ homotopies $F : [0,1] \times [0,1] \longrightarrow X$ and $G : [0,1] \times [0,1] \longrightarrow X$ such that $F(s,0) = f_0(s)$, $F(s,1) = f_1(s) = G(s,0)$, $F(0,t) = x_0 = G(0,t)$, $F(1,t) = x_1 = G(1,t)$ and $G(s,1) = f_2(s)$. Define

$$H(s,t) = \begin{cases} F(s,2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\ G(s,2t-1) & \text{if } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

Then, $H(s,0) = F(s,0) = f_0(s)$, $H(s,1) = G(s,1) = f_2(s)$, $H(0,t) = F(0,2t) = x_0$ if $0 \leqslant t \leqslant \frac{1}{2}$ and $H(0,t) = G(0,2t-1) = x_0$if $\frac{1}{2} \leqslant t \leqslant 1$ whereas $H(1,t) = F(1,2t) = x_1$ if $0 \leqslant t \leqslant \frac{1}{2}$ and $H(1,t) = G(1,2t-1) = x_1$ if $\frac{1}{2} \leqslant t \leqslant 1$. For a visual representation, see Figure 5. ■

This suggests that we can also concatenate homotopies horizontally. There is an interesting method: let $x_0, x_1, x_2 \in X$ and let f be a path from x_0 to x_1 and g be a path from x_1 to x_2. Define $f.g : [0,1] \longrightarrow X$ by

$$(f.g)(s) = \begin{cases} f(2s) & \text{if } 0 \leqslant s \leqslant \frac{1}{2} \\ g(2s-1) & \text{if } \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

Then, $f.g$ is a path from x_0 to x_1. Clearly, $(f.g)(0) = f(0) = x_0$ and $(f.g)(1) = g(1) = x_2$. At $s = \frac{1}{2}$, we have $f(1) = g(0) = x_1$. Notice that x_0, x_1, x_2 do not have to be distinct. We could have loops around the point, like petals.

Lemma 16 *If f_0 and f_1 are homotopic paths between x_0 and x_1, g_0 and g_1 are homotopic paths between x_1 and x_2, then $f_0.g_0$ is homotopic to $f_1.g_1$. Moreover, $f_1.g_0 \simeq f_0.g_1$*

Proof. Let $F : [0,1] \times [0,1] \longrightarrow X$ be the homotopy between f_0 and f_1 and $G : [0,1] \times [0,1] \longrightarrow X$ be the homotopy between g_0 and g_1. Define $H(s,t) = (f_t.g_t)(s)$.

$$H(s,t) = (f_t.g_t)(s) = \begin{cases} f_t(2s) & \text{if } 0 \leqslant s \leqslant \frac{1}{2} \\ g_t(2s-1) & \text{if } \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

If f_0 is a path between x_0 and x_1 and g_0 is path between x_1 and x_2, then we have that $H(s,0) = f_0.g_0$ is path between x_0 and x_2. Similarly for $H(s,1) = f_1.g_1$. Also, $H(0,t) = f_t(0) = x_0$ and $H(1,t) = g_t(1) = x_2$. Thus, H is the required homotopy. For $f_1.g_0 \simeq f_0.g_1$, we can have $H'(s,t) = (f_{1-t}.g_t)(s)$. Then, $H(s,0) = f_1.g_0$ and $H(s,1) = f_0.g_1$ and $H(0,t) = f_{1-t}(0) = x_0$ and $H(1,t) = g_t(1) = x_2$. ■

This lemma shows that we can define concatenation on homotopy classes: if $[f]$ and $[g]$ are homotopy classes in which the terminal of $[f]$ is the initial of $[g]$, then we can have $[f].[g] = [f.g]$.

And now, we can say what a fundamental group actually is. Let $x_0 \in X$, where X is a topological space. Then, the **fundamental group** of X based at x_0 is $\pi_1(X,x_0) = \{[f] : f \text{ is a path from } x_0 \text{ to } x_0\} = \{[f] : f \text{ is a loop based on } x_0\}$ with binary operation of concatenation (as above). This is a group.

Proof. It is clear that concatenation is a binary operation. To prove associativity, suppose $[f],[g],[h] \in \pi_1(X,x_0)$. We essentially need to prove that $[f].([g].[h]) \simeq ([f].[g]).[h]$. For the left side, the time scale of the representative f would be half whereas that for g,h would both be a quarter. Let its path be P_1. For the right side, the time scale of h would be half whereas that for f,g would both be a quarter. Let its path be P_2. Define $\phi : [0,1] \longrightarrow [0,1]$ by

$$\phi(s) = \begin{cases} 2s & 0 \leqslant s \leqslant \frac{1}{4} \\ s + \frac{1}{4} & \frac{1}{4} \leqslant s \leqslant \frac{1}{2} \\ \frac{1}{2}s + \frac{1}{2} & \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

Then, $P_2(s) = P_1(\phi(s))$. Now define $\phi_t(s) = (1-t)s + t\phi(s)$. Note that $\phi_0(s) = id_I$ whereas $\phi_1(s) = \phi(s)$. And now, we can define $H(s,t) = P_1(\phi_t(s))$. Then, $H(s,0) = P_1(\phi_0(s)) = P_1(s)$, $H(s,1) = P_1(\phi_1(s)) = P_1(\phi(s)) = P_2(s)$ and $H(0,t) = P_1(\phi_t(0)) = P_1(t\phi(0)) = P_1(0) = x_0$ and $H(1,t) = P_1(\phi_t(1)) = P_1(1-t+t\phi(1)) = P_1(1-t+t) = P_1(1) = x_1$.

Thus, the binary operation is associative. The identity for this group is $e : [0,1] \longrightarrow X$ is the constant map $e(x) = x_0$ for all x. To show this, let f be a loop based at x_0. We need to show that $e.f \simeq f$ and $f.e \simeq f$.

If $e.f$ is a path P_1 and f is a path P_2, then time interval for P_1 can be split by half, each for e and f and, as we deform to f, the half reserved for f becomes a full unit interval. Let $\phi : [0,1] \longrightarrow [0,1]$ be defined by

$$\phi(s) = \begin{cases} 0 & \text{if } 0 \leqslant s \leqslant \frac{1}{2} \\ 2s-1 & \text{if } \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

Then, $P_1(s) = P_2(\phi(s))$. Define $\phi_t(s) = (1-t)s + t\phi(s)$ and then use this to define homotopy $H : [0,1] \times [0,1] \longrightarrow X$ by $H(s,t) = P_2(\phi_t(s))$. Then,

$H(s,0) = P_2(\phi_0(s)) = P_2(s)$, $H(s,1) = P_2(\phi_1(s)) = P_2(\phi(s)) = P_1(s)$ and $H(0,t) = P_2(\phi_t(0)) = P_2(t\phi(0)) = P_2(0) = x_0$ whereas $H(1,t) = P_2(\phi_t(1)) = P_2(1-t+t\phi(1)) = P_2(1-t+t) = P_2(1) = x_0$.

To show that $f.e \simeq f$, let P_1 be the path for the right side (with the interval divided into two, one half reserved for f and other for e) and P_2 be the path for the right side. Let $\phi : [0,1] \longrightarrow [0,1]$ be defined by

$$\phi(s) = \begin{cases} 2s & \text{if } 0 \leqslant s \leqslant \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

Then, $P_1(s) = P_2(\phi(s))$. Now, define $\phi_t(s) = (1-t)s + t\phi(s)$ and then use this to define homotopy $H : I \times I \longrightarrow X$ by $H(s,t) = P_2(\phi_t(s))$. Then, $H(s,0) = P_2(\phi_0(s)) = P_2(s)$, $H(s,1) = P_2(\phi_1(s)) = P_2(\phi(s)) = P_1(s)$ and $H(0,t) = P_2(\phi_t(0)) = P_2(t\phi(0)) = P_2(0) = x_0$ whereas $H(1,t) = P_2(\phi_t(1)) = P_2(1-t+t\phi(1)) = P_2(1) = x_0$

Every element f in $\pi_1(X,x_0)$ has an inverse g. That is, $\forall f, \exists g : [f].[g] = [e] = [g].[f]$. That is, $f.g \simeq e \simeq g.f$. Here is how we construct it: define $g(s) = f(1-s)$. For $f.g$, this actually moves the point x_0 along f from the end towards somewhere in the middle s_0 and then reverses direction. This happens for each $s_0 \in [0,1]$. For $g.f$, the directions are reversed but the idea is the same. In both cases, we might be moving x_0 along f in one direction or another but we end up with a path that is homotopic to the constant function e. For this, use the homotopy

$$H(s,t) = (f_t.g_t)(s) = \begin{cases} f_t(2s) & \text{if } 0 \leqslant s \leqslant \frac{1}{2} \\ g_t(2s-1) & \text{if } \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

where $f_t : [0,1] \longrightarrow X$ is loop based at x_0 such that

$$f_t(s) = \begin{cases} f(s) & \text{if } 0 \leqslant s \leqslant 1-t \\ g(s) & \text{if } 1-t \leqslant s \leqslant 1 \end{cases}$$

and $g_t(s) = f(1-t)$ for all s. Then, $H(0,t) = f_t(0) = f(0) = x_0$ if $0 \leqslant s \leqslant 1-t$ and $= g_t(1) = f(1-t) = x_1$ if $1-t \leqslant s \leqslant 1$. $H(1,t) = g_t(1) = f(1-t)$, $H(s,0) = (f_0.g_0)(s) = f_0(s) = f(s)$ and $H(s,1) = (f_1.g_1)(s) = g(s) = f(1-t)$ if $0 \leqslant s \leqslant \frac{1}{2}$ and, again, $g_1(2s-1) = f(1-t)$ if $\frac{1}{2} \leqslant s \leqslant 1$ ■

The 1 in the subscript reminds us that we can view I as S^1 (because we're forcing end-points to agree). This can be generalised to S^n.

Example 17 *Let $X = \mathbb{R}^k$ and let $x_0 \in X$. We've seen that, for any two points in X, any path between these two points are homotopic. Thus, $\pi_1(\mathbb{R}^k, x_0) = \{[e]\}$.*

For path connected spaces, there is an interesting result which assures us that the choice of base point is invariant

Theorem 18 *If X is path connected and x_0, x_1 are points in X, then $\pi_1(X,x_0) \simeq \pi_1(X,x_1)$. That is, the fundamental groups are isomorphic.*

Proof. The proof is interesting because it is constructive, not done by contradiction.

Let $f \in \pi_1(X, x_1)$. Then, $[h].[f].[h]^{-1} \in \pi_1(X, x_0)$ where h is a path between x_0 and x_1. Same argument as the inverses in the proof for group above applies, except that over there, we could have said that the intersection point is x_0, which did not matter.

Denote $[h]^{-1} = [\bar{h}]$ and define $\beta_h : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$ by $\beta_h([f]) = [h.f.\bar{h}]$. To show that this function is well-defined, we need to show that $f \simeq f' \implies h.f.\bar{h} \simeq h.f'.\bar{h}$. Since $\bar{h} \simeq \bar{h}$, $f \simeq f'$ and $h \simeq h$ and concatenation respects homotopy, so the function is well-defined.

To show that it is a homomorphism, note that $\beta_h([f.g]) = [h.f.g.\bar{h}] = [h.f.\bar{h}.h.g.\bar{h}] = [h.f.\bar{h}].[h.g.\bar{h}] = \beta_h(f)\beta_h(g)$.

$\beta_{\bar{h}} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ is the inverse homomorphism. ∎

In usual literature, a space (X, τ) is said to be simply connected if it is path connected and if any two paths are homotopic. With what we have, we can rephrase this definition as follows:

Definition 19 *If a space (X, τ) is path connected and for some $x_0 \in X$, $\pi_1(X, x_0) = \{[e]\} \simeq 1$, then X is said to be **simply connected**.*

That is, instead of saying "for some", we can say "for any".

Recall that if a space is path connected, then it is connected. However, if the space X was disconnected, let (U, V) be a separation of X. Then, in particular, X is not path connected. Without loss of generality, we can assume that U is a connected component containing x_0 and will, therefore, contain a path connected component. From what we have seen above, $\pi_1(X, x_0) = \pi_1(U \cup V, x_0) \simeq \pi_1(U, x_0)$. Thus, such groups only see the path component containing x_0.

Theorem 20 *If X is path connected, then X is simply connected if and only if there is a unique homotopy class of paths between any pair of points.*

Proof. Let $x_0, x_1 \in X$ and X be simply connected. Then, in particular, it is path connected and so, $\pi_1(X, x_0) \simeq \{[e]\} \simeq \pi_1(X, x_1)$. Let f and g be paths from x_0 to x_1 and let e_0 (respectively, e_1) be the constant path starting at x_0 (respectively, x_1). Note that $f.g^{-1} \simeq e_0$ and that $g.g^{-1} \simeq e_1$. Then, $f \simeq f.e_1 \simeq f.(g^{-1}.g) \simeq (f.g^{-1}).g \simeq e_0.g \simeq g$.

Conversely, let $x_0 \in X$, then if there is any homotopy class of paths between any two points, then if f is a loop based at x_0, then $f \simeq e$ so $\pi_1(X, x_0) \simeq \{[e]\}$. ∎

3.1 Fundamental group of S^1

Our first big theorem says that $\pi_1(S^1) \simeq \mathbb{Z}$. Notice the absence of the base-point. This is because S^1 is path connected.

3.1.1 Foray into Covering Spaces

The approach will be by first embedding S^1 in $\mathbb{R}^2$. We can conveniently choose any point in S^1, as seen in $\mathbb{R}^2$, so we can let $x_0 = (1,0)$. Now, consider a loop $w : [0,1] \longrightarrow S^1$, defined by $w(s) = (\cos 2\pi s, \sin 2\pi s)$. We will use this to prove that $\pi_1(S^1, x_0) = \langle w \rangle$. That is, if $n \in \mathbb{Z}$, then $[w]^n = [w_n]$ where $w_n = (\cos 2n\pi s, \sin 2n\pi s)$. To accomplish this, we need to show that (a) if $\gamma : [0,1] \longrightarrow S^1$ is a loop based at x_0, then $\gamma \simeq w_n$ for some $n \in \mathbb{Z}$. That is, $[w]$ is a generator for $\pi_1(S^1, x_0)$ and (b) to show that the order of w is infinite, we will show that if $w_n \simeq w_m$, then $m = n$. Thus, we can show that $\varphi : \mathbb{Z} \longrightarrow \pi_1(S^1, x_0)$, defined by $n \longmapsto [w]^n$, is an isomorphism.

In order to do (a) and (b), we need to define $p : \mathbb{R} \longrightarrow S^1$ by $p(s) = (\cos 2n\pi s, \sin 2n\pi s)$. This map is an instance of what's called a covering space.

Definition 21 *Let (X, τ) be a topological space. A* ***covering space*** *of X is a topological space $\widetilde{X}$ and a map $p : \widetilde{X} \longrightarrow X$ such that, $\forall x \in X$, there is a neighborhood V containing x such that $p^{-1}(V)$ is a disjoint union of open sets, each of which is mapped homeomorphically by p to V.*

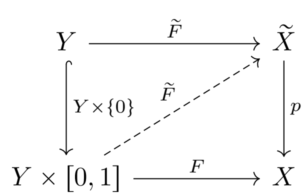
V is an **evenly covered neighborhood**. The map p is usually called the covering projection, or sometimes the covering map.

Lemma 22 (Homotopy Lifting Lemma) *Let $p : \widetilde{X} \longrightarrow X$ be a covering map. Suppose that there are maps $F : Y \times [0,1] \longrightarrow X$ and $\widetilde{F} : Y \times \{0\} \longrightarrow \widetilde{X}$, so that*

$$p \circ \widetilde{F}\Big|_{Y \times \{0\}} = F|_{Y \times \{0\}}$$

Then there is a unique extended map $\widetilde{F} : Y \times [0,1] \longrightarrow \widetilde{X}$, lifting F (such that $p \circ \widetilde{F} = F$)

In other words, given a homotopy and a lift at the left end point, there is a unique lift of F that extends to $\widetilde{F}$. The following diagram depicts the situation:



Proof. We first claim that for each $y_0 \in Y$, there is a neighborhood N containing y_0 on which we can build a lift of F. In other words, there is a map $\widetilde{F} : N \times [0,1] \longrightarrow \widetilde{X}$ so that $p \circ \widetilde{F} = F|_{N \times [0,1]}$.

To prove this, for each $t \in [0,1]$, we can pick an evenly covered neighborhood U_t of $F(y_0, t) \in X$. By continuity, there is a product neighborhood $N_t \times (a_t, b_t)$ where N_t is an open subset of Y and (a_t, b_t) is an open subset of $[0,1]$ such that $F(N_t \times (a_t, b_t)) \subset U_t$. By compactness, $\{y_0\} \times [0,1]$ can be covered by finitely many neighborhoods of this type and so, we can choose a single (connected) neighborhood $N \subset Y$ such that $y_0 \in N$ and a finite partition $0 = t_0 < t_1 < ... < t_n = 1$ so that $F(N \times [t_i, t_{i+1}]) \subset U_i$.

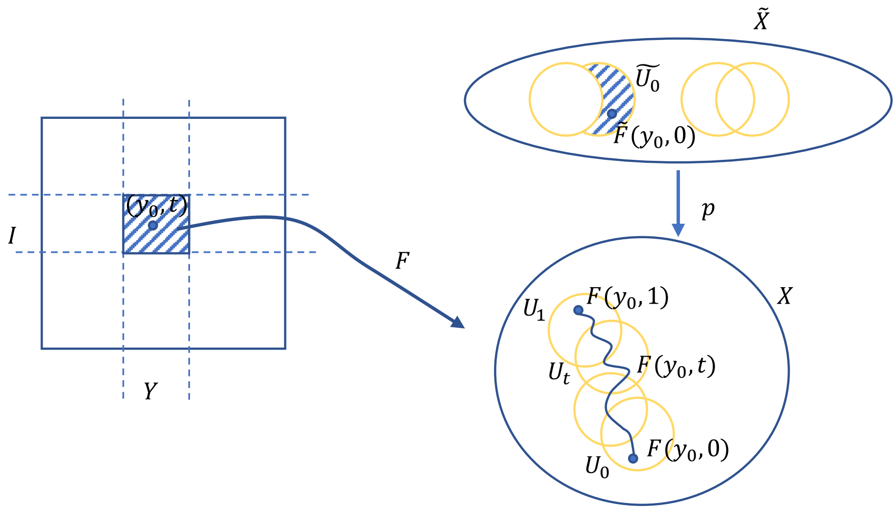


Figure 6: *Choice of N_t*

Now, we will lift by building $\widetilde{f} : N \times [0,1] \longrightarrow \widetilde{X}$ by induction. Let $\widetilde{U}_0$ be the component of $p^{-1}(U_o)$ that contains $\widetilde{F}(y_0, 0)$. This set is a homeomorphic copy of U_0. For $(y,t) \in N \times [0, t_1]$, we can define $\widetilde{F}(y,t) = p^{-1}|_{U_0}(F(y,t))$. Assume that we have defined $\widetilde{F}$ on $N \times [0, t_i]$. We know that $F(N \times [t_i, t_{i+1}]) \subset U_i$. Let $\widetilde{U}_i$ be the lift of U_i that contains $\widetilde{F}(y_0, t_i)$. Define for $(y,t) \in N \times [t_i, t_{i+1}]$, let $\widetilde{F}(y,t) = p^{-1}|_{U_i}(F(y,t))$

Thus, for each $y_0 \in Y$, there is a neighborhood N of y_0 on which we can build $\widetilde{F} : N \times [0,1] \longrightarrow \widetilde{X}$ by $\widetilde{F}(y,t) = p^{-1}(F(y,t))$

We now need to show that this lifting in unique. That is, if $y_0 \in Y$ and $\widetilde{F}_i : [0,1] \longrightarrow \widetilde{X}$, so that

$$p\left(\widetilde{F}_i(t)\right) = F(y_0, t)$$

for $i = 1, 2$ and $F_2(0) = F_1(0)$, then $\widetilde{F}_1 = \widetilde{F}_2$.

We can again use the standard compactness argument. As before, pick $0 = t_0 < t_1 < ... < t_n = 1$ so that $F(\{y_0\} \times [t_i, t_{i+1}]) \subset U_i$ (evenly covered neighborhood). Since $[t_0, t_1]$ is connected, $\widetilde{F}_i([t_0, t_1])$ must be contained in a single lift (connected images under continuous maps are connected) $\widetilde{U}_0$ of U_0. Since $\widetilde{F}_1(0) = \widetilde{F}_2(0)$, it follows that $\widetilde{U}_0$ is independent of i. Since the projection p is injective on $\widetilde{U}_0$ and $p\widetilde{F}_1 = p\widetilde{F}_2$, it follows that

$$\widetilde{F}_1\Big|_{[t_0, t_1]} = \widetilde{F}_2\Big|_{[t_0, t_1]}$$

Now, we repeat the argument on each component of the partition of t. Assume by induction that

$$\widetilde{F}_1\Big|_{[t_0,t_i]} = \widetilde{F}_2\Big|_{[t_0,t_i]}$$

Since the interval $[t_i, t_{i+1}]$ is connected, there is a unique lift $\widetilde{U}_i$ of U_i so that $F_i([t_i, t_{+1}]) \subset \widetilde{U}_i$. p is injective on $\widetilde{U}_i$ and $p\widetilde{F}_1 = p\widetilde{F}_2$ and so

$$\widetilde{F}_1\Big|_{[t_0,t_{i+1}]} = \widetilde{F}_2\Big|_{[t_0,t_{i+1}]}$$

And we're done with uniqueness. The next claim is as follows: if $N_1, N_2 \subset Y$ are open sets and we can build lifts $\widetilde{F}_1 : N_1 \times [0,1] \longrightarrow \widetilde{X}$ and $\widetilde{F}_2 : N_2 \times [0,1] \longrightarrow \widetilde{X}$ of F_1, then we can build a lift $\widetilde{F} : (N_1 \cup N_2) \times [0,1] \longrightarrow \widetilde{X}$. To see this, let

$$\widetilde{F}(y,t) = \begin{cases} \widetilde{F}_1(y,t) & \text{if } y \in N_1 \\ \widetilde{F}_2(y,t) & \text{if } y \in N_2 \end{cases}$$

This function is continuous on the union, provided that we can prove the function is well-defined on the intersection $N_1 \cap N_2$. If the intersection is disjoint, there is nothing to check. If it isn't, we use the previous result: let $y \in N_1 \cap N_2$. By construction, $\widetilde{F}_1(y,0) = \widetilde{F}_2(y,0)$ and by previous part of this proof, $\widetilde{F}_1(y,t) = \widetilde{F}_2(y,t)$ for all t, so $\widetilde{F}$ is well-defined on the intersection.

In summary, for each $y \in Y$, we can find $N_y \ni y$ and a lift $\widetilde{F} : N_y \times [0,1] \longrightarrow \widetilde{X}$. By the previous claim, we can assemble F_y into $F : Y \times [0,1] \longrightarrow \widetilde{X}$ by $F(z,t) = F_y(z,t)$ if $z \in N_y$. Now, if $\widetilde{F}_1$ and $\widetilde{F}_2$ are both lifts of F, then by uniqueness, $\widetilde{F}_1 = \widetilde{F}_2$, since they agree on sets of the form $\{y_0\} \times [0,1]$. ■

Theorem 23 *Let $p : \widetilde{X} \longrightarrow X$ be a covering space. Then,*

1. *$\forall$ paths $f : [0,1] \longrightarrow X$ starting at x_0 and for each $\widetilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\widetilde{f} : [0,1] \longrightarrow \widetilde{X}$ starting at $\widetilde{x}_0$. In other words, all paths can be lifted uniquely*

2. *For each homotopy of paths (homotopy rel. to end points) $F : [0,1] \times [0,1] \longrightarrow X$ starting at x_0 and for each $\widetilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\widetilde{F} : [0,1] \times [0,1] \longrightarrow \widetilde{X}$ starting at $\widetilde{x}_0$*

Proof. The proof is a direct consequence of the **Homotopy Lifting Lemma**. For 1, use $Y = \{pt\}$ and the fact that $\{pt\} \times [0,1] \simeq [0,1]$. For 2, let $Y = [0,1]$ and by 1, there is a unique lift $F : \widetilde{Y} \times \{0\} \longrightarrow \widetilde{X}$ starting at $\widetilde{x}_0$. By **Homotopy Lifting Lemma**, we can uniquely extend $\widetilde{F}$ to $Y \times [0,1]$. Since F is a homotopy of paths, $F(0,t)$ is constant and so, $\widetilde{F}$, the lifted extension, maps $\widetilde{F}(0,t)$ to $\widetilde{x}_0$. ■

Now, to prove the fact that the fundamental group of S^1 is $\mathbb{Z}$, first, note that $\mathbb{R}$ is a covering space for S^1 via the map $p : \mathbb{R} \longrightarrow S^1$ defined by $p(x) = (\cos 2\pi x, \sin 2\pi x)$.

Proof. Let U be an interval in S^1. That is, we can have $2k\pi + \theta' < x < 2k\pi + \theta$ for $k \in \mathbb{Z}$. For each k, $2k\pi + \theta' < x < 2k\pi + \theta$ is disjoint from k'. ■

With this knowledge, we need to ask ourselves whether there is a map $\widetilde{f} : X \longrightarrow \mathbb{R}$ so that $p \circ \widetilde{f} = f$. By the way, if such map exists, then we say that $\widetilde{f}$ **is the lift of** f with respect to p. Since we have $w_n : [0,1] \longrightarrow S^1$ and $p : \mathbb{R} \longrightarrow S^1$, we can have $\widetilde{w}_n(s) = ns$. Then, $p \circ \widetilde{w}_n(s) = p(ns) = w_n(s)$. In fact, even $\widetilde{w}_n(s) = ns + k$ works for any k.

Now, we need to show that if $\gamma : [0,1] \longrightarrow S^1$ is a loop based at x_0, then $\gamma \simeq w_n$ for some $n \in \mathbb{Z}$. This will show that $[w]$ is a generator for $\pi_1(S^1, x_0)$

Proof. Let $f : [0,1] \longrightarrow S^1$ be a loop at x_0. Then, by 1 of **Theorem 23**, there is a lift $\widetilde{f} : [0,1] \longrightarrow \widetilde{X}$ starting at 0 and so $\widetilde{f}(1) = n$ for some $n \in \mathbb{Z}$ because $\widetilde{f}$ is a loop. We do know that all paths in $\mathbb{R}$ with same end points are homotopic. Thus, $\widetilde{f}$ is homotopic (rel. end points) to the path $\widetilde{w}_n$ which starts at 0 and ends at n. So, there is a homotopy $\widetilde{F} : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ from $\widetilde{f}$ to $\widetilde{w}_n$ and so $F = p \circ \widetilde{F}$ is a homotopy between f and w_n. ■

And now, we show that the order of w is infinite. We do this by showing that if $w_n \simeq w_m$, then $m = n$.

Proof. Let $F : [0,1] \times [0,1] \longrightarrow S^1$ be homotopy between w_n and w_m. Then, by 2 of **Theorem 23**, we can lift F to $\widetilde{F} : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ starting at 0. By uniqueness of 1 of **Theorem 23**, $\widetilde{f_0} = \widetilde{F}(s,0) = \widetilde{w}_n$. Similarly, $\widetilde{f_1} = \widetilde{f}(s,1) = \widetilde{w}_m$. Since F is a homotopy, it is continuous and $\mathbb{Z} \subset \mathbb{R}$ is discrete, it follows that $\widetilde{F}(1,t)$ is constant and $\widetilde{F}(1,0) = n$ and $\widetilde{F}(1,1) = m$, thus $m = n$. ■

Theorem 24 *We now prove that $\pi_1(S^n)$ for $n \geqslant 2$ is trivial.*

Proof. Note that $S^n \backslash \{x\}$ is homotopy equivalent to R^n for any point $x \in S^n$. To see this, we can imagine pulling $S^2 \backslash \{x\}$ from where x is and then stretching the remainder of S^2 flat out. This would cover entire $\mathbb{R}^2$. Arguments works for a general n. Since $\mathbb{R}^n$ is simply-connected, this implies that $S^n \backslash \{x\}$ is simply-connected. So if we show that any loop f in S^n is homotopic to some loop g in $S^n \backslash \{x\}$, then f will be nullhomotopic since g is in a simply-connected space.

To show $f \simeq g$ for some g, consider some point $x \in S^n$ that is not the basepoint of f. Let N be a neighborhood of x. Since f is continuous, $f^{-1}(N)$ is open in the relative topology $[0,1]$. Since $f^{-1}(N)$ is open, we can pick intervals (a_i, b_i) such that each $f(a_i, b_i)$ is pathconnected in N and with $\partial \overline{N} = \{f(a_i), f(b_i)\}$ and

$$f^{-1}(N) = \bigcup_{i \in I} (a_i, b_i)$$

for some indexing set I. Since f is continuous and $\{x\}$ is compact in N, $f^{-1}(\{x\})$ is compact in $f^{-1}(N)$. Thus, there is a finite subcover of $f^{-1}(\{x\})$ such that

$$f^{-1}(\{x\}) = \bigcup_{k=1}^{n} (a_i, b_i)$$

Let $f_i : [a_i, b_i] \longrightarrow \overline{N}$ be the path segment of f corresponding to $[a_i, b_i]$ and let $g_i : [a_i, b_i] \longrightarrow \{f(a_i), f(b_i)\}$ be the path segment of f corresponding to

$[a_i, b_i]$. By construction, $f_i \simeq g_i$ with $g_i(x) \neq f(x)$ on $Int\left(\overline{N}\right)$. Form the path g by replacing all the f_i with g_i in f, and note that $f \simeq g$ and g does not cross x on $Int\left(\overline{N}\right)$. ■

3.1.2 Applications

What can we prove with the machinery we have so far? The fundamental theorem of algebra!

Theorem 25 *Every non-constant polynomial has a root in* $\mathbb{C}$.

The idea of the proof as is follows: the function $z \longmapsto z^n$ takes the complex plane and covers it around n-times. This is called the complex covering. If there was a polynomial with no roots, then the covering of the unit circle would get reduced to a trivial loop, a contradiction.

Proof. Let $p(z)$ be a non-constant, monic polynomial $p(z) = z^n + a_1 z^{n-1} + \ldots + a_n$, for $n \geqslant 1$ with no roots. Since p has no roots, we can define, for $r \geqslant 0$, let

$$f_r(s) = \frac{p\left(re^{2\pi is}\right)/p(r)}{\left|p\left(re^{2\pi is}\right)/p(r)\right|}$$

This function pays attention to what the polynomial does on the circle of radius r. This is a homotopy of loops in S^1 based at 1. When $r = 0$, we get the constant loop $f_0(s) = 1$ so for any R, $[f_R(s)] = 0 \in \pi_1\left(S^1\right)$. For now, let $R = \max\left(|a_1| + |a_2| + \ldots + |a_n|, 1\right)$. Then, if $|z| = R$, then $R \geqslant 1$ and so $\left|z^{n-1}\right| \geqslant \left|z^{n-2}\right|$. Thus, $|z^n| \geqslant (|a_1| + |a_2| + \ldots + |a_n|)\left|z^{n-1}\right| \geqslant |a_1|\left|z^{n-1}\right| + |a_2|\left|z^{n-2}\right| + \ldots + |a_n| \geqslant \left|a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n\right|$. Let $P_t(s) = z^n + t\left(a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n\right)$. Then, $P_t(z)$ has no roots on the circle $|z| = R$ for $0 \leqslant t \leqslant 1$. Replacing p with P_t in $f_R(s)$, we get

$$f_{R,t}(s) = \frac{P_t\left(Re^{2\pi is}\right)/P_t(R)}{\left|P_t\left(Re^{2\pi is}\right)/P_t(R)\right|}$$

This is a homotopy of loops. Note that $f_{R,1}(s) = f_R(s)$ and

$$f_{R,0}(s) = \frac{r^n e^{2\pi nis}/r^n}{\left|r^n e^{2\pi nis}/r^n\right|} = (\cos 2\pi ns, \sin 2\pi ns) = w_n(s)$$

so $0 \simeq f_R \simeq w_n$ so $n = 0$. That is, the polynomial is constant, a contradiction. ■

We can also prove Brower's Fixed Point Theorem.

Theorem 26 *If $f : D^2 \longrightarrow D^2$ is continuous, then there is a point $x \in D^2$ such that $f(x) = x$.*

Proof. Let $h : D^2 \longrightarrow D^2$ be a continuous map with no fixed points. We can construct a function $r : D^2 \longrightarrow S^1$ defined by connecting $h(x)$ and x on the boundary of D^2. That is, there is a unique $t > 0$ such that $|(1-t)h(x) + tx| =$

1. Let $r(x) = (1-t)h(x)+tx$. By implicit function theorem, $r(x)$ is continuous and so, if $x \in S^1$, then $r(x) = x$. Since $\partial D^2 = S^1$, r is a retraction. But there cannot be any retraction from D^2 to S^1: suppose that f_0 is a loop in S^1. If f_0 is a loop in D^2, then there is a homotopy f_t between f_0 and a constant loop so that $g_t = r \circ f_t$ is a homotopy between $r \circ f_0 = f_0$ and a constant loop in S^1. Thus, $[f_0] = 0 \in \pi_1(S^1)$, a contradiction. ■

For the next theorem, imagine S^2 embedded in $\mathbb{R}^3$. Antipodes are then diametrically opposite points.

Theorem 27 (Borsak-Ulam) *Let $n = 2$. Then, for every continuous map $f : S^n \longrightarrow \mathbb{R}^n$, $\exists$ a pair of antipodes $x, -x$ so that $f(x) = f(-x)$*

Proof. Assume that for every continuous map $f : S^2 \longrightarrow \mathbb{R}^2$, $f(x) \neq f(-x)$ $\forall x \in S^2$. Then, define $g : S^2 \longrightarrow S^1$ by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

and then define a loop $\eta : [0,1] \longrightarrow S^2$, given by $\eta(s) = (\cos 2\pi s, \sin 2\pi s, 0)$, a loop which goes on the equator of S^2. η is nullhomotopic since $\pi_1(S^2) \simeq \{[e]\}$. We can now define another loop $h : [0,1] \longrightarrow S^1$ defined as $g \circ \eta$. Since η is nullhomopotic, h is also nullhomotopic. Note that

$$g(-x) = \frac{f(-x) - f(x)}{|f(x) - f(-x)|} = -\frac{f(x) - f(-x)}{|f(x) - f(-x)|} = -g(x)$$

so that g is an odd function. Moreover, $\eta\left(s + \frac{1}{2}\right) = (\cos(2\pi s + \pi), \sin(2\pi s + \pi), 0)$
$= (\cos 2\pi s \cos \pi - \sin 2\pi s \sin \pi, \sin 2\pi s \cos \pi + \sin \pi \cos 2\pi s, 0)$
$= (-\cos 2\pi s, -\sin 2\pi s, 0) = -\eta(s)$ and that $h\left(s + \frac{1}{2}\right) = g\left(\eta\left(s + \frac{1}{2}\right)\right) = g(-\eta(s)) = -g(\eta(s)) = -h(s)$. Now, lift the loop h to $\mathbb{R}$: let $\widetilde{h}$ be the lift of h that starts at 0. Then, $\widetilde{h}\left(s + \frac{1}{2}\right) = \widetilde{h}(s) + \frac{q_s}{2}$ where q_s is an odd integer. This integer may depend on s but the mapping $s \longmapsto q_s$ is continuous and hence constant. Let $q_s = q$. How far apart are $\widetilde{h}(0)$ and $\widetilde{h}(1)$? Put $s = \frac{1}{2}$ to get $\widetilde{h}(1) = \widetilde{h}\left(\frac{1}{2}\right) + \frac{q}{2} = \widetilde{h}(0) + \frac{q}{2} + \frac{q}{2} = \widetilde{h}(0) + q$, so $h \simeq w_q$, but q is odd and so, $h \neq 0$. That is, the loop is simultaneously homotopic to the constant path and not. ■

In fact, the Borsak-Ulam theorem is true for any dimension. Thus, there are always opposite points on the earth where temperature and pressure are both the same.

Alternatively, we can equivalently say that every continuous $f : S^2 \longrightarrow \mathbb{R}^2$ is non-injective. This is without reference to embedding.

Imagine a tangent plane on the sphere with orthogonal lines projecting to the sphere. Then, we can always find antipodal points.

Corollary 28 *There is no embedding from S^2 to $\mathbb{R}^2$*

A cool application of this is that if S^n can be decomposed into $n+1$ closed subsets, then at least one of them contains a pair of antipodes.
Proof. Let A_1, A_2 and A_3 be closed sets. Define $d_i : S^2 \longrightarrow \mathbb{R}$ by $d_i(x) = d(x, A_i)$, the distance between x and A_i. This is a continuous function. Now set $f : S^2 \longrightarrow \mathbb{R}^2$ $f = (d_1, d_2)$. f is a continuous function so that there are antipodes x_0 and $-x_0$ so that $f(x) = f(-x)$. That is, they $d(x_0, A_i) = d(-x_0, A_i)$ for $i = 1, 2$. If $d_i(x_0) = 0$ for $i = 1, 2$, then x_0 and $-x_0$ are in A_i. If $d_1(x_0) \neq 0 \neq d_2(x_0)$, so $x_0, -x_0 \notin A_1, A_2$ so that $x_0, -x_0 \in A_3$. ■

4 The Functor

We won't enter in to the details of the functor which produces the fundamental group but we will test waters by specifically seeing products and morphisms in topological spaces and their corresponding fundamental groups.

4.1 Products

Here's a start:

Theorem 29 *If X and Y are path connected topological spaces, then the fundamental group $\pi_1(X \times_{\mathcal{T}op} Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times_{\mathcal{G}rp} \pi_1(Y, y_0)$*

Proof. Recall that for $f : Z \longrightarrow X \times Y$, we can have $f = (g, h)$ with $g : Z \longrightarrow X$ and $h : Z \longrightarrow Y$ and that f is continuous iff g and h are continuous. This tells us that each loop f based at a point (x_0, y_0) is equivalent to loops g in X based at x_0 and h in Y based at y_0. The same is true for homotopies: each homotopy $f_t : Z \longrightarrow X \times Y$ is equivalent to a pair of homotopies $f_t = (g_t, h_t)$ with $g_t : Z \longrightarrow X$ and $h_t : Z \longrightarrow Y$. Now, we can define a homomorphism $\phi : \pi_1(X \times_{\mathcal{T}op} Y, (x_0, y_0)) \longrightarrow \pi(X, x_0) \times_{\mathcal{G}rp} \pi(Y, y_0)$ by $\phi([f]) = ([g], [h])$. This is a bijection, clearly, provided that the function is well-defined.

It *is* well-defined: let $f_0 \simeq f_1$ (paths in $X \times Y$) and let $f_t : [0,1] \longrightarrow X \times Y$ be the corresponding homotopy. Then, we can have $f_0 = (g_0, h_0)$ and $f_1 = (g_1, h_1)$. By the second part of the reasoning above, there are homotopies $g_t : [0,1] \longrightarrow X$ (between g_0 and g_1) and $h_t : [0,1] \longrightarrow Y$ (between h_0 and h_1). That is, $(g_0, h_0) \simeq (g_1, h_1)$.

Now, to show that ϕ is a homomorphism, notice that if $f_0 = (g_0, h_0)$ and $f_1 = (g_1, h_1)$, then $f_0.f_1 = (g_0.g_1, h_0.h_1)$ so that $\phi([f_0] . [f_1]) = \phi([f_0.f_1])$
$= ([g_0.g_1], [h_0.h_1]) = ([g_0], [h_0]) * ([g_1], [h_1]) = \phi([f_0]) * \phi([f_1])$, where $*$ is natural binary operation on $\pi(X, x_0) \times_{\mathcal{G}rp} \pi(Y, y_0)$. ■

An important example is as follows: the example of a manifold, the tori. The n-torus, $T^n := S^1 \times S^1 \times ... \times S^1$ (n copies). Then, $\pi_1(S^1 \times S^1 \times ... \times S^1) \simeq \mathbb{Z}^n$. To clarify, imagine the usual torus, T^2. Any pair of integers then tells us how many times one needs to go about both circles, which make up T^2. The trefoil knot is a $(3,2)$ curve on the torus (see Figure 7).

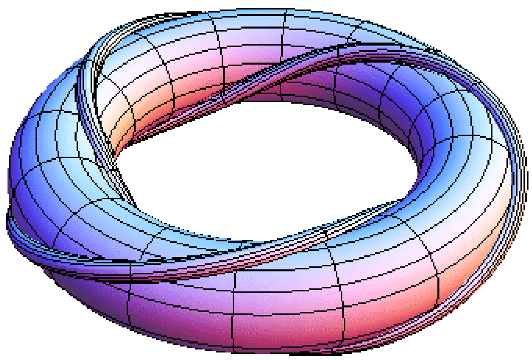


Figure 7: *Image taken from stackexchange*

4.2 Induced Homomorphisms

The idea is that continuous maps between topological spaces induce a homomorphism between their corresponding fundamental groups.

Let X and Y be topological spaces with base points x_0 and y_0. Let $\varphi : X \longrightarrow Y$ be continuous with $\varphi(x_0) = y_0$. We can denote this as continuous map between pointed topological spaces $\varphi : (X, x_0) \longrightarrow (Y, y_0)$. Now, if $[f] \in \pi_1(X, x_0)$, then $[\varphi \circ f] \in \pi_1(Y, y_0)$. $\varphi \circ f$ is called a **pushforward** of f. This actually gives us a homomorphism φ_* between the corresponding fundamental groups, defined by $\varphi_*([f]) = [\varphi \circ f]$, where $\varphi_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$. This function is well-defined homomorphism.

Proof. To show that φ_* is well-defined, let $[f_0], [f_1] \in \pi_1(X, x_0)$ with $[f_0] = [f_1]$. Then, $f_0 \simeq f_1$. Let f_t be a homotopy between f_0 and f_1. That is, $f_t(s) : X \times [0,1] \longrightarrow X$ is continuous map. Since φ is continuous, then $\varphi \circ f_t$ is continuous as the composition of continuous maps is continuous. Furthermore, $\varphi \circ f_t$ is clearly a homotopy between $\varphi \circ f_0$ and $\varphi \circ f_1$.

Let $[f], [g] \in \pi_1(X, x_0)$. Concatenate their push-forwards (why?):

$$\varphi \circ (f.g) = (\varphi \circ f) . (\varphi \circ g) = \begin{cases} \varphi \circ f(2s) & \text{if } 0 \leqslant s \leqslant \frac{1}{2} \\ \varphi \circ g(2s-1) & \text{if } \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

so that $\varphi_*([f] . [g]) = \varphi_*([f.g]) = [\varphi \circ (f.g)] = [(\varphi \circ f) . (\varphi \circ g)]$
$= [(\varphi \circ f)] . [(\varphi \circ g)] = \varphi_*([f]) . \varphi_*([g])$ ■

Lemma 30 *If* $id : (X, \tau_X) \longrightarrow (X, \tau_X)$ *is the identity homeomorphism, then* $\varphi_* : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_0)$ *is the identity homomorphism. Furthermore, if* $\varphi : X \longrightarrow Y$ *and* $\psi : Y \longrightarrow Z$ *are two continuous maps with* $\varphi(x_0) = y_0$ *and* $\psi(y_0) = z_0$ *and* $\varphi_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ *and* $\psi_* : \pi_1(Y, y_0) \longrightarrow \pi_1(Z, z_0)$ *the induced homomorphisms, then the induced homomorphism* $(\psi \circ \varphi)_*$ *of* $\psi \circ \varphi$ *is equal to* $\psi_* \circ \varphi_*$*. That is,* $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$

Proof. The first is easy to see: $\varphi_*([f]) := [id_X \circ f] = [f]$.

For the second, let $[f] \in \pi_1(X, x_0)$. Then, $(\psi \circ \varphi)_* ([f]) = [(\psi \circ \varphi) \circ f] = [\psi \circ (\varphi \circ f)] = \psi_* ([\varphi \circ f]) = \psi_* \circ \varphi_* ([f])$ for all $[f]$. ■

That is, π_1 induces a covariant functor from the category of pointed topological spaces to the category of groups. A corollary of this is that π_1 is homeomorphism invariant.

Corollary 31 *If $\varphi : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ is a homeomorphism, then $\varphi_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is an isomorphism*

Proof. It suffices to prove that $\left(\varphi^{-1}\right)_* = (\varphi_*)^{-1}$:

$\left(\varphi \circ \varphi^{-1}\right)_* = \varphi_* \circ \left(\varphi^{-1}\right)_* = id_* = \left(\varphi^{-1}\right)_* \circ \varphi_* = \left(\varphi^{-1} \circ \varphi\right)_*$ ■

How are these related to retractions? The following theorem provides an answer. Recall that if (X, τ_X) is a topological space and $A \subset X$, then a retraction from X to A is a map $r : (X, \tau_X) \longrightarrow (A, \tau_A)$ such that $r \circ i = id_A$.

Theorem 32 *If (X, τ_X) retracts onto a subspace (A, τ_A), then $i_* : \pi_1(A, x_0) \longrightarrow \pi_1(X, x_0)$ is an injection homomorphism. If (X, τ_X) deformation retracts onto (A, τ_A), then i_* is an isomorphism.*

Proof. Let $r : (X, \tau_X) \longrightarrow (A, \tau_A)$ be our retraction. Then, $r \circ i = id_A$ and so, $r_* \circ i_* = (id_A)_*$. Thus, i_* is an injection.

If (X, τ_X) deformation retracts to (A, τ_A), then there is a homotopy $r_t : (X, \tau_X) \longrightarrow (X, \tau_X)$ such that $r_0 = id_X$, r_1 is a retraction onto (A, τ_A) and $r_t|_A = id_A$ $\forall t$. Let $f : [0,1] \longrightarrow X$ be a loop based at $x_0 \in A$. Then, $r_1 \circ f$ is a loop in A based at x_0 that is homotopic to f by $r_t \circ f$ because $r_0 \circ f = id_X \circ f = f$. It follows that $i_* ([r_1 \circ f]) = [r_1 \circ f] = [f]$. So, i_* is surjective. ■

What is the group theoretic analog of a retraction? It's a projection: let $H \leqslant G$ and $p : G \longrightarrow H$ such that $p|_H = id_H$ (i.e. $p^2 = p$), so p is surjective and hence we get the a split short exact sequence

$$\{[e]\} \longrightarrow K = \ker p \hookrightarrow G \overset{p}{\rightleftarrows} H \longrightarrow \{[e]\}$$

where $H \simeq G/K$. In other words, $G \simeq K \rtimes H$, the semidirect product of K and H. If H is normal, then $G \simeq K \times H$.

This also tells us that there is no retraction $r : D^2 \longrightarrow S^1$ because $\pi_1\left(D^2\right) = \{[e]\}$, $\pi_1\left(S^1\right) = \mathbb{Z}$ and there is no injection from $i_* : \pi_1\left(S^1\right) \longrightarrow \pi_1\left(D^2\right)$.

The invariance of the fundamental groups also follows from homotopy equivalence. That is, if $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ is a homotopy equivalence, then $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is an isomorphism.

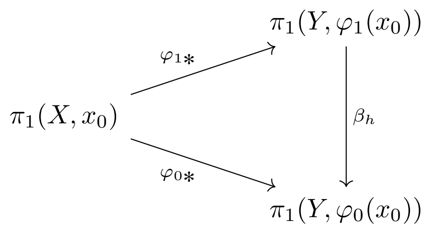
The idea of the proof is simple but there is one detail which needs consideration: since we know that we have a $g : (Y, \tau_Y) \longrightarrow (X, \tau_X)$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$, we might be tempted to use $g_* : \pi_1(Y, y_0) \longrightarrow \pi_1(X, x_0)$. However, note that $\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, gf(x_0))$, so we have a base point issue. Simply put, homotopic maps always don't induce the same homomorphism. We can get around this by building a path between the two base points.

Definition 33 *Let $h_t : (X, x_0) \longrightarrow (Y, y_0)$ be a homotopy of maps. If $h_t(x_0) = y_0 \forall t$, then h_t is said to be **base-point preserving.***

Proposition 34 *Let $h_t : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ be a base-point preserving homotopy. Then, $(h_0)_* = (h_1)_*$.*

Proof. $(h_0)_*([f]) = [h_0 \circ f] = [h_1 \circ f] = (h_1)_*([f])$. Since this holds $\forall [f] \in \pi_1(X, x_0)$, we are done (where do we use the fact that h_t is base-point preserving?) ■

Lemma 35 *Let $\varphi_t : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ be a homotopy and $h : [0,1] \longrightarrow (Y, \tau_Y)$ be a path, given by $h(s) = \varphi_s(x_0)$. Then, $\beta_h \circ (\varphi_1)_* = (\varphi_0)_*$ i.e. the following diagram commutes:*



Proof.

Let $h_t(s) = h(st)$. This traces the path h but in timescale $[0,t]$. Let f be a loop in X based at x_0. Then, $g_t = h_t.(\varphi_t \circ f).\overline{h}_t$ is a loop based at $\varphi_0(x_0)$ homotopic to $\varphi_0 \circ f$. This is shown in Figure 8. As t varies, each loop $h_t.(\varphi_t \circ f).\overline{h}_t$ moves and is then based at $\varphi_t(x_0)$.

If $t = 1$, then $h_1.(\varphi_1 \circ f).\overline{h}_1 \simeq \varphi_0 \circ f$. Using this, we arrive at $\beta_h \circ (\varphi_1)_*([f])$

$= \beta_h([\varphi_1 \circ f])$

$= \beta_h([\varphi_0 \circ f])$

$= \left[h_t \circ (\varphi_t \circ f) \circ \overline{h}_t\right]$

$= [\varphi_0 \circ f]$

$= \left((\varphi_0)_*\right)([f])$. ■

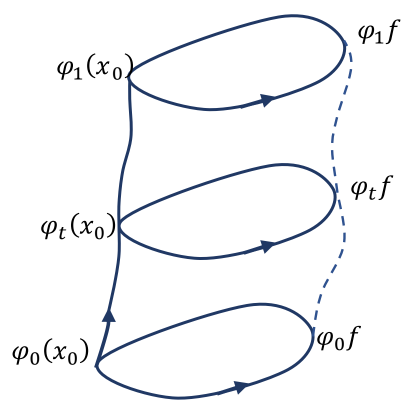


Figure 8: *Loops g_t based at $\varphi_t(x_0)$*

Theorem 36 *If $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ is a homotopy equivalence, then $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is an isomorphism.*

Proof. $\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, gf(x_0)) \xrightarrow{f'_*} \pi_1(Y, g(f(f(x_0))))$, where we have f'_* different from f_* because the domains are different. Since $g \circ f \simeq id_X$, so by the above lemma, there is a path h such that $g_* \circ f_* = \beta_h \circ (id_X)_* = \beta_h$, so $g_* \circ f_*$ is an isomorphism. Hence $g_* \circ f_*$ is a bijection and so, f_* is injective and g_* is surjective. Similarly, $f_* \circ g_*$ is also an isomorphism and so, g_* is also injective and f_* is surjective. ■

5 Van Kampen's Theorems

And now, we will explore a tool to compute fundamental groups. The idea is to compute $\pi_1(X)$ by breaking X into simpler spaces, then use these fundamental groups to compute $\pi_1(X)$.

As an alternative to **Theorem 24**, here is a precurosor: if $n \geqslant 2$, then $\pi_1(S^n) = \{[e]\}$. This is because an n-sphere can be built by taking two n-discs and gluing them along their boundary, i.e. S^{n-1} and because every path γ in S^n can be decomposed (up to homotopy) as a product $\gamma_1.\gamma_2.....\gamma_n \simeq \gamma$, where each γ_i is in one of the disks.

Recall free products: assume that we have two groups G and H and we want a group Γ which contains both G and H. We could get away with $\Gamma = G \times H$ but then we'd have $G \leqslant C_\Gamma(H)$ and, conversely, $H \leqslant C_\Gamma(G)$, so we're adding relations we might not need. This is where the free product comes.

Let $\{A_\alpha\}_{\alpha \in C}$ be a collection of path connected (open) topological spaces. Let

$$x_0 \in \bigcap_{\alpha \in C} A_\alpha$$

For each $\alpha \in C$, we have

$$i_\alpha : A_\alpha \hookrightarrow X = \bigcup_{\alpha \in C} A_\alpha$$

and an induced map $(i_\alpha)_* : \pi_1(A_\alpha, x_0) \hookrightarrow \pi_1(X, x_0)$. By the universal properties of free products, we get

$$\Phi : \mathop{\ast}_{\alpha \in C} \pi_1(A_\alpha, x_0) \longrightarrow \pi_1(X, x_0).$$

Theorem 37 (VKT-1) *If $\{A_\alpha\}_{\alpha \in C}$ is a collection of open, path connected sets, each containing a point x_0 and let*

$$X = \bigcup_{\alpha \in C} A_\alpha$$

If for each $\alpha, \beta \in C$, $A_\alpha \cap A_\beta$ is path connected (assuming that the empty set is path connected), then

$$\Phi : \mathop{\ast}_{\alpha \in C} \pi_1(A_\alpha, x_0) \longrightarrow \pi_1(X, x_0).$$

is a surjection.

Proof. Let $\gamma : [0,1] \longrightarrow X$ be a loop in X based at x_0. Our idea will be find $\gamma_1, \gamma_2, ..., \gamma_k$ so that γ_i is a loop in A_{α_i}. These assumptions will essentially help us conclude that each loop γ in X based at x_0 decomposes up to homotopy as $\gamma_1.\gamma_2.....\gamma_k$ where each $\gamma_i \subset A_{\alpha_i}$. That is, $\gamma \simeq \gamma_1.\gamma_2.....\gamma_k$ so that $\Phi([\gamma_1].[\gamma_2].....[\gamma_k]) = [\gamma]$, which can help us with the proof.

Let $C_\alpha = \gamma^{-1}(A_\alpha)$, then $\{C_\alpha\}_{\alpha \in C}$ is a cover of I. By Lebesgue Number Lemma, there is a partition of I, $0 = t_0 < t_1 < ... < t_n = 1$ so that $\gamma([t_i, t_{i+1}]) \subset$

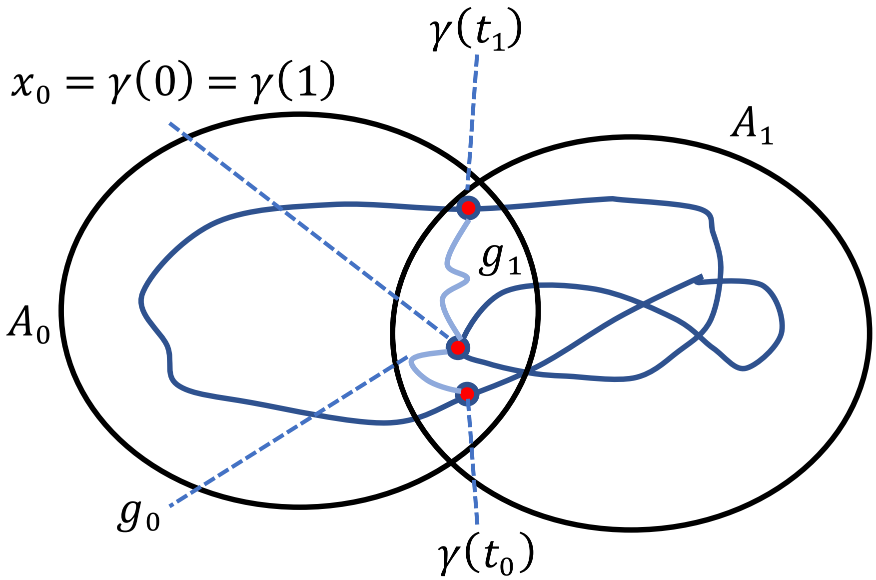


Figure 9: *Decomposition of loops*

A_{α_i} for some $\alpha_i \in C$ (see Figure 9). Let $\gamma_i = \gamma([t_i, t_{i+1}])$. By hypothesis, $A_{\alpha_i} \cap A_{\alpha_{i+1}}$ is path connected and so, there is a path g_i from x_0 to $\gamma_i(t_{i+1})$. Then, $\gamma \simeq \left(\gamma_0 g_0^{-1}\right)\left(g_0\gamma_1 g_1^{-1}\right)\ldots\left(g_{n-2}\gamma_{n-1}\right)$ where $\gamma_0 g_0^{-1} \in A_{\alpha_0}$, $g_0\gamma_1 g_1^{-1} \in A_{\alpha_1}$ and $g_{n-2}\gamma_{n-1} \in A_{\alpha_{n-1}}$. ■

Going back to S^n for $n \geqslant 2$, we again show that $\pi_1(S^n) \simeq \{[e]\}$ in light of VKT-1: for two open, path connected pieces A_1 and A_2 of S^n, $A_1 \cap A_2 \simeq S^{n-1} \times \mathbb{R}$ so that $\Phi : \pi_1(A_1, x_0) * \pi_1(A_2, x_0) \longrightarrow \pi_1(S^n, x_0)$ gives us a surjection of two trivial groups.

Another consequence of VKT-1 is that, if $n \neq 2$, then $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^2$. This is called the invariance of domain.

Proof. If $n = 1$, look at past homework. For $n > 2$, if $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^n$ is a homeomorphism. Then, $\mathbb{R}^2 \backslash \{0\} \simeq \mathbb{R}^n \backslash \{\varphi(0)\}$. Since $\mathbb{R}^2 \backslash \{0\} \simeq S^1 \times \mathbb{R}$. The latter is an infinite cylinder. To visualize this, imagine pushing from the punctured point outward towards a unit circle and then rotating each point outside this unit circle continuously. This, therefore, tells us that, in general, $\mathbb{R}^n \backslash \{0\} \simeq S^{n-1} \times \mathbb{R}$. But $\pi_1\left(S^1 \times \mathbb{R}\right) \simeq \mathbb{Z}$ and $\pi_1\left(S^{n-1} \times \mathbb{R}\right) \simeq \pi_1\left(S^{n-1}\right) \times \pi_1(\mathbb{R}) \simeq \{[e]\}$ since $n \geqslant 3$, a contradiction. ■

From the first isomorphism theorem, we have

$$\pi_1(X, x_0) \simeq \mathop{\ast}_{\alpha \in C} \pi_1(A_\alpha, x_0) / \ker \Phi$$

What is the kernel? That comes from van Kampen's Second Theorem, VKT-2. Before we look into the statement of the theorem and the proof, we look at an example.

Consider a genus 2 torus. Split it into two open, path connected sets A_α and A_β, as shown in Figure 10.

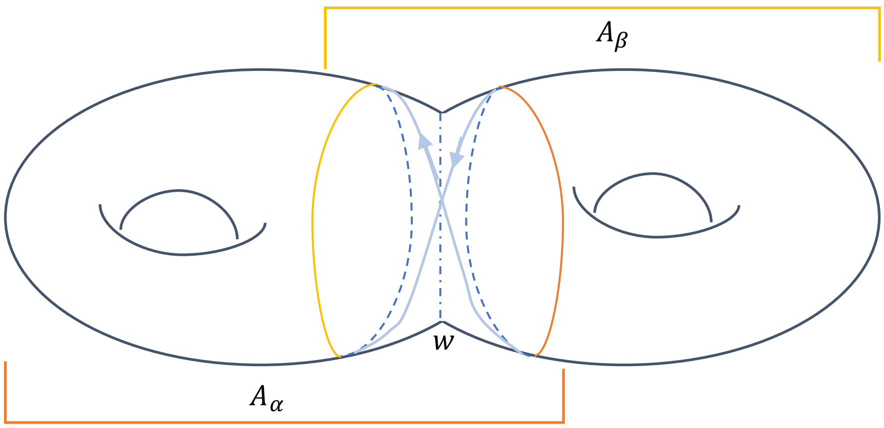


Figure 10: *Decomposition of genus 2 torus*

Note that, for α, β, $A_\alpha \cap A_\beta$ is path connected and that we have inclusion maps $i_{\alpha\beta} : A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ and $i_{\beta\alpha} : A_\alpha \cap A_\beta \hookrightarrow A_\beta$ for free. With these, we get corresponding inclusion induced homomorphisms

$$\pi_1\left(A_\alpha \cap A_\beta, x_0\right) \xhookrightarrow{(i_{\alpha\beta})_*} \pi_1\left(A_\alpha, x_0\right) \xhookrightarrow{(i_\alpha)_*} \pi_1\left(X, x_0\right)$$

$$\pi_1\left(A_\alpha \cap A_\beta, x_0\right) \xhookrightarrow{(i_{\beta\alpha})_*} \pi_1\left(A_\beta, x_0\right) \xhookrightarrow{(i_\beta)_*} \pi_1\left(X, x_0\right)$$

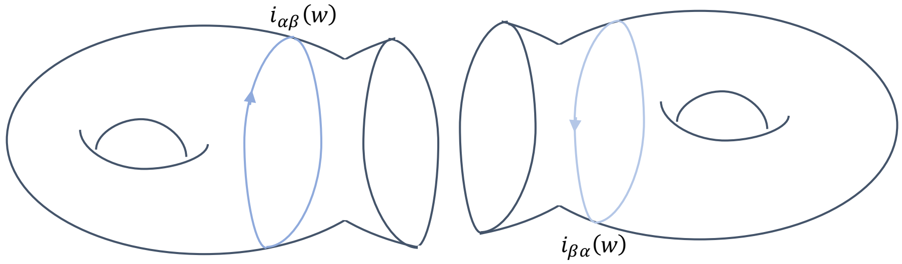
and the diagram commutes. Let $w \in \pi_1\left(A_\alpha \cap A_\beta, x_0\right)$. Since $i_\alpha \circ i_{\alpha\beta} = i_\beta \circ i_{\beta\alpha}$, we must have $i_\alpha i_{\alpha\beta}(w) = i_\beta i_{\beta\alpha}(w)$ and so $i_\alpha i_{\alpha\beta}(w)\left(i_\beta i_{\beta\alpha}(w)\right)^{-1} = e$. Let $\tau = i_{\alpha\beta}(w)\left(i_{\beta\alpha}(w)\right)^{-1}$. Then, $\Phi(\tau) = i_\alpha i_{\alpha\beta}(w)\, i_\beta\left(\left(i_{\beta\alpha}(w)\right)^{-1}\right) = i_\alpha\left(i_{\alpha\beta}(w)\right) i_\beta\left(\left(i_{\beta\alpha}(w)\right)^{-1}\right) = e$ and so $\tau \in \ker\Phi$ (see Figure 11). Note that in each set, the loops are trivial but not in the free product.

This gives us a group

$$N = \left\langle i_\alpha\left(i_{\alpha\beta}(w)\right) i_\beta\left(\left(i_{\beta\alpha}(w)\right)^{-1}\right) : \alpha, \beta \in C, w \in \pi_1\left(A_\alpha \cap A_\beta, x_0\right)\right\rangle \subset \ker\Phi$$

Theorem 38 (VKT-2) *If $\{A_\alpha\}_{\alpha \in C}$ is a collection of open, path connected sets, each containing a point x_0 and let*

$$X = \bigcup_{\alpha \in C} A_\alpha$$



(a) A_α **(b)** A_β *(inverse here?)*

Figure 11: *Decomposed parts and corresponding paths*

If for each $\alpha, \beta, \gamma \in C$, $A_\alpha \cap A_\gamma \cap A_\beta$ is path connected (assuming that the empty set is path connected), then

$$\Phi : \underset{\alpha \in C}{*} \pi_1 (A_\alpha, x_0) \longrightarrow \pi_1 (X, x_0)$$

is a surjection with kernel

$$N = \left\langle (i_{\alpha\beta}(w)) (i_{\beta\alpha}(w))^{-1} : \alpha, \beta \in C, w \in \pi_1 (A_\alpha \cap A_\beta, x_0) \right\rangle$$

The trick here is to use $i_\alpha : \pi_1 (A_\alpha, x_0) \hookrightarrow \pi_1 (X, x_0)$.

Proof. If $[f] \in \pi_1 (X, x_0)$, then a factorization of $[f]$ is a product $[f_1][f_2] ... [f_n]$ where f_i is a loop in some A_{α_i} based at x_0 and $[f_i] \in \pi_1 (A_{\alpha_i}, x_0)$ and $f \simeq f_1.f_2....f_n$. This can be thought of as a road way to get surjection. Since the map Φ is not necessarily injective, we may have many factorizations corresponding to different loops. To remedy this, we can give an equivalence relation: two factorizations of $[f]$ will be called equivalent if they differ by a sequence of the following

1. If $[f_i], [f_{i+1}] \in \pi_1 (A_{\alpha_i}, x_0)$, then we can replace with $[f_i.f_{i+1}]$
2. If $[f_i] \in A_\alpha \cap A_\beta$ is a loop, then regard $[f]$ as an element of $\pi_1 (A_\alpha, x_0)$ instead of $\pi_1 (A_\beta, x_0)$ (this isn't allowed in free products of groups in general)

Such an equivalence relation doesn't change the elements of

$$\frac{* \pi_1 (A_\alpha, x_0)}{N}$$

To show this, let $[f_i] \in \pi_1 (A_\alpha \cap A_\beta, x_0)$. Let $[f_i^\alpha] = i_{\alpha\beta} ([f_i]) \in \pi_1 (A_{\alpha_i}, x_0)$ and $\left[f_i^\beta\right] = i_{\beta\alpha} ([f_i]) \in \pi_1 (A_{\beta_i}, x_0)$. Then, $[f_1][f_2] ... [f_i][f_{i+1}] ... [f_n]$

$$= [f_1][f_2] ... \left[f_i^\beta\right] [f_i^\alpha]^{-1} [f_i^\alpha] [f_{i+1}] ... [f_n]$$

$= [f_1][f_2] \ldots \left(\left[f_i^{\beta}\right][f_i^{\alpha}]^{-1}\right)[f_i^{\alpha}][f_{i+1}] \ldots [f_n]$
$= [f_1][f_2] \ldots [e][f_i^{\alpha}][f_{i+1}] \ldots [f_n]$
$= [f_1][f_2] \ldots [f_i^{\alpha}][f_{i+1}] \ldots [f_n]$

The goal is then to show that all the factorizations of $[f]$ are equivalent. This is equivalent to showing that

$$\widetilde{\Phi} : \left(\mathop{*}_{\alpha \in C} \pi_1 (A_\alpha, x_0)\right) / N \longrightarrow \pi_1 (X, x_0)$$

is an isomorphism.

Suppose that $[f_1] \ldots [f_k]$ and $[f_1'] \ldots [f_l']$ are factorizations of $[f] \in \pi_1 (X, x_0)$. Then, $f_1.f_2....f_k \simeq f_1'.f_2'....f_l'$. Let $F : [0,1] \times [0,1] \longrightarrow X$ be homotopies between them. By compactness, choose partitions $0 = s_0 < s_1 < \ldots < s_m = 1$ and $0 = t_0 < t_1 < \ldots < t_n = 1$ compatible with the factorizations so that $F([s_{i-1}, s_i] \times [t_{j-1}, t_j]) = R_{ij} \subset A_{\alpha_{ij}}$. Each intersection might will be in four boxes, then! Perturb one grid (see Figure 12).

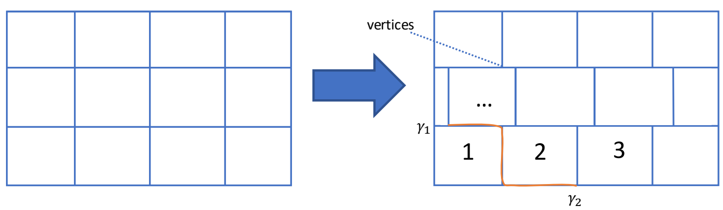


Figure 12: *Perturbation of vertices*

Number the bricks. Now, one point lives in at most three bricks. Call one corner of a brick a vertex. Let γ_i be the path which separates the first i bricks from the rest. Observe that we can get a factorization for $F|_{\gamma_r}$ by picking paths in the appropriate intersections from the images of vertices back to the basepoint. Thus, going from γ_0 to γ_1 changes the path but doesn't change the homotopy class of the first piece of the factorization because it is happening in same piece A_i. ■

5.1 Wedge Sums

Another useful tool is called the wedge sum. Let $\{(X_\alpha, p_\alpha)\}_{\alpha \in C}$ be a collection of pointed topological spaces. Let $\bigvee_{\alpha \in C} X_\alpha$ be the quotient obtained by identifying all p_α to a single point p. If $\{(X_\alpha, p_\alpha)\}_{\alpha \in C}$ are loops, then we get $G_{|C|}$, a rose with $|C|$ petals. Using this, we can compute $\pi_1 \left(\bigvee_{\alpha \in C} X_\alpha, p\right)$. Suppose that for each $p_\alpha \in X_\alpha$, there is a neighborhood U_α of p_α in X_α that deformation retracts to p_α. Then, $\pi_1 \left(\bigvee_{\alpha \in C} X_\alpha, p\right) \simeq \mathop{*}_{\alpha \in C} \pi_1 (X_\alpha, p_\alpha)$. In this case, X_α is a deformation retract of $X_\alpha \bigvee_{\beta \neq \alpha} U_\beta = A_\alpha$ (say). Note that the intersection of two or more A_α is $\bigvee_{\alpha\in C} U_\alpha$, and hence contractible. This gives us the isomorphism $\Phi : \mathop{\ast}_{\alpha\in C} \pi_1 (X_\alpha, p_\alpha) \longrightarrow \pi_1 \left(\bigvee_{\alpha\in C} X_\alpha, p\right)$.

In short, we have just proved that $\pi_1 \left(\bigvee_{\alpha\in C} S^1\right) \simeq \mathop{\ast}_{\alpha\in C} \mathbb{Z}$ and gotten rid of the promised Euler characteristic.

Theorem 39 *Let (X_α, p_α) be a collection of path connected topological spaces. If each p_α has a contractible neighborhood U_α, then $\pi_1 \left(\bigvee_{\alpha\in C} (X_\alpha, p_\alpha)\right) \simeq \mathop{\ast}_{\alpha\in C} \pi_1 (X_\alpha, p_\alpha)$*

Proof. Let $A_\alpha = X_\alpha \bigvee_{\alpha\neq\beta} U_\beta$. This is path connected and satisfy VKT-2. Then, we get the surjection $\Phi : \mathop{\ast}_{\alpha\in C} \pi_1 (X_\alpha, p_\alpha) \longrightarrow \pi_1 \left(\bigvee_{\alpha\in C} X_\alpha, p\right)$ where $\ker \Phi$ is trivial. ■

Another example: consider a square with all vertices connected (i.e. a square with two diagonals). Compute the maximal tree. Label the un-traversed edges e_1, e_2, e_3 and traversed path T. Shrink it down to a point. Take an ϵ neighborhood of $T \cup e_i$. Deformation retract to $T \cup e_i$. This is a circle (see picture) The fundamental group of these guys is $\mathbb{Z}$. Note: $T \cup e_i$ intersected over i deformation retracts to T so fundamental group of the intersection of these guys is trivial and so, fundamental group of the graph is $\mathbb{Z}^3$

Another graph. The theta graph G (looks like a rotated, squashed theta). It's the above but with one edge removed. Again, we get same T but this time, only e_1 and e_2. Label two three vertices a, b, c and let $X_a = G\setminus\{a\}$, $X_b = G\setminus\{b\}$ and $X_c = G\setminus\{c\}$. These are path connected as is the intersection of any two of them but intersection of all three isn't. Note that $\pi_1 (X_i) \simeq \mathbb{Z}$ and $\pi_1 (X_i \cap X_j)$ for $i \neq j$ is trivial because the intersection is contractible. This also contradicts VKT-2 because if we did get the surjection $\Phi : \pi_1 (X_a) * \pi_1 (X_b) * \pi_1 (X_c) \longrightarrow \pi_1 (G)$. The kernel is trivial. Hence we get two copies of $\mathbb{Z}$ isomorphic to three, a contradiction.

Here is a politically charged example: the Hawaiin Earring. Let C_n be a circle of radius $\frac{1}{n}$ centred at $(1/n, 0)$. Let

$$X = \bigcup_n C_n$$

The space X is not the wedge of the earrings: the fundamental group of the wedge of the cirlces is isomorphic to countable copies of $\mathbb{Z}$ whereas the fundamental group of X is uncountable. For each n, let $r_n : X \longrightarrow C_n$. Combine these maps get a surjection

$$p : \pi_1 (X) \longrightarrow \Pi_n \mathbb{Z}$$

on the space of all integer valued sequences. This space is uncountable.

To show that p is a surjection, let $(k_1, k_2, ...) \in \Pi_n \mathbb{Z}$. Build a loop based at p that loops k_n times around the n-th circle on the interval $\left[1 - \frac{1}{n}, 1 - \frac{1}{n+1}\right]$. That is, for $k_1 = 3$, p loops around 3 times and C_1 from $\left[0, \frac{1}{2}\right]$. This loop is continuous path on $[0, 1)$. The original is going to contain all but finitely many circles. That is, for $n \geqslant N$, $\left[1 - \frac{1}{n+1}, 1\right] \subset U(\epsilon; 0)$. Thus, p is a surjection. p is not an isomorphism since $\pi_1(X)$ is not abelian. Let $p_n : \pi_1(X) \longrightarrow *\mathbb{Z}$. A non-abelian group cannot surject on to an abelian group.

5.2 Lens Spaces

These are 3-manifolds. That is, locally homeomorphic. Every point locally looks like $\mathbb{R}^3$. $N_2 = D^2 \times S^1$, the solid torus and $N_1 = S^1 \times S^1$, the usual torus. Both are not 3-manifolds, hence not Lens spaces because any point on N_2 does not give balls.

Let $f : \partial N_1 \longrightarrow \partial N_2$. This is a homeomorphism. Let $L_f = N_1 \sqcup N_2 / \sim$ where $x \sim y$ if $x \in \partial N_1, y \in \partial N_2$ such that $f(x) = y$. L_f is a 3-manifold.

Let $A_i = N_i^{\epsilon}$, the ϵ neighborhood of N_i in L_f. Then, $\pi_1(A_1 \cap A_2) \simeq \mathbb{Z}^2$ because $A_1 \cap A_2 \simeq T^2 \times I$

By VKT,

$$\begin{aligned} \pi_1(L_f) &\simeq \frac{\pi_1(A_1) * \pi_1(A_2)}{\left\langle i_{12}(w)(i_{21}(w))^{-1} : w \in \pi_1(A_1 \cap A_2)\right\rangle} \\ &\simeq \langle [w_1][w_2] : i_{12}(w) = i_{21}(w)\, \forall w \in \pi_1(A_1 \cap A_2)\rangle \end{aligned}$$

Observe a curve w_1 on the donut (see picture) can be homotoped into $A_1 \cap A_2$. Thus, in $\pi_1(L_f)$, $[w_1] = [w_2]^k$ for some $k \in \mathbb{Z}$, making the group $\pi_1(L_f)$ is cyclic. Let m_1, m_2 be arbitrary curves in N_1 and N_2 with $m_2 = f(m_1)$. Then, $[m_1]$ is trivial in $\pi_1(N_1)$ and hence trivial in $\pi_1(L_f)$. Note that $i_{12}[m] = [m_1] \in \pi_1(L_f)$ and $[m_2] = [w_1]^m = C'_{21}[m_1]$ for some $m \in \mathbb{Z}\backslash\{0\}$ and that $\{[e]\} = [m_1] = [w_2]^m$. Thus, $\pi_1(L_f) \simeq \langle [w_2] : [w_2]^m = 1\rangle \simeq \mathbb{Z}_m$. If $m = 0$, the curve is mapped to a meridian, giving us a cyclic group of infinite order.

L_f is the Lens space. Different homeomorphisms give us different Lens Spaces. Certain homeomorphisms might give us homeomorphic Lens spaces, which we will see in due time.

5.2.1 Classification of Lens Spaces

Given $f_1, f_2 : T^2 \longrightarrow T^2$, when are L_f and L'_f homeomorphic? That is, homotopy equivalent.

Again, $T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$. What is the action of $SL_2(\mathbb{Z})$. Note that $SL_2(\mathbb{Z})$ acts on $\mathbb{R}^2$ and preserves integer vectors. That is, the action of $SL_2(\mathbb{Z})$ on $\mathbb{Z}^2$ gives us $\mathbb{Z}^2$. All homeomorphisms arise from $M : T^2 \longrightarrow T^2$, where $M \in SL_2(\mathbb{Z})$.

For the previous example, let $\{m_1, w_1\}$ be basis for first turns and let $\{m_2, w_2\}$ be basis for second turns. Then,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = Mm_1 = \begin{pmatrix} a \\ c \end{pmatrix} = a\,[m_2] + c\,[w_2]$$

where $[m_2]$ is the meridian and $[w_2]$ is the longitude.

Theorem 40 *Let*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } M' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

be matrices in $SL_2(\mathbb{Z})$. Then, L_M is homeomorphic to $L_{M'}$ if and only if $c = c'$ and $a = \pm a' \, mod c$. L_M is homotopy equivalent to $L_{M'}$ if and only if $c = c'$ and $\pm aa'$ is a square in $\mathbb{Z}/c\mathbb{Z}$

Thus, we can denote $L_M = L(c, a)$ since b and d play no role. Then, $L(7,1) \simeq L(7,6)$ and $L(7,2) \simeq L(7,4) \simeq L(7,3) \simeq L(7,5)$

$L(1,0) \simeq S^3$ because $S^3 \backslash N_1 \simeq N_2$. $L(0,1) \simeq S^2 \times S^1$ and $L(2,1) \simeq \mathbb{R}P^3$ (think of $B^3/x \sim -x$. Drill a hole, say N_2 (twisted torus). The remainder is another torus. Paths move and come back.. hence closed curves!

5.3 CW Complexes again

Let X be a path connected space and let Y be the space obtained from X by gluing a collection $\{e_\alpha\}_{\alpha \in C}$ of 2-cells to X using attaching maps $\varphi_\alpha : S^1 \longrightarrow X$. Our goal is to compute $\pi_1(Y)$ in terms of $\pi_1(X)$ and attaching maps, where $S^1 = \partial e_\alpha$

A disc with two holes X, with fundamental group isomorphic to $\mathbb{Z}^2$, add nipples to the holes. Path $\gamma_1.\varphi_1.\overline{\gamma_1}$ is null homotopic in Y whereas $\gamma_2.\varphi_2.\overline{\gamma_2}$ is not (see picture).

In general, for any base point s_0, in the domain of the attaching map S^1, let $x_\alpha = \varphi_\alpha(s_0)$ and let γ_α be a path in X from x_0 to x_α. Then, $\gamma_\alpha.\varphi_\alpha.\overline{\gamma_\alpha}$ is null homotopic in Y and the subgroup $N = \langle\langle \gamma_\alpha.\varphi_\alpha.\overline{\gamma_\alpha} : \alpha \in C \rangle\rangle \leqslant \pi_1(X)$ normally generated by $\gamma_\alpha.\varphi_\alpha.\overline{\gamma_\alpha}$ can give us a natural homomorphism $i_* : \pi_1(X, x_0) \hookrightarrow \pi_1(Y, y_0)$. We will show that i_* is a surjection with kernel N.

In order to show this, we prove a more general theorem.

Theorem 41

1. *If Y is obtained from X by attaching a collection $\{e_\alpha\}_{\alpha \in C}$ of 2-cells, then the inclusion $i_* : \pi_1(X, x_0) \hookrightarrow \pi_1(Y, y_0)$ is a surjection with kernel N.*

2. *If Y is obtained from X by attaching a collection of n-cells $\{e_\alpha\}_{\alpha \in C}$ with $n > 2$, then $\pi_1(X) \simeq \pi_1(Y)$*

3. *If X is a CW Complex, then $\pi_1(X) \simeq \pi_1\left(X^{(2)}\right)$*

What this says is that CW Complexes are low-dimensional tools. From 2 implies 3, we see that CW Complexes forget higher dimensional attachment of n-cells.

Proof. Let Z be the space obtained from Y by attaching rectangles $S_\alpha \simeq [0,1]\times[0,1]$ so that $[0,1]\times\{0\}$ is attached along γ_α and $\{1\}\times[0,1]$ is attached at x_α along a "radius" of e_α. Z deformation retracts onto Y. Thus, $\pi_1(Z)\simeq\pi_1(X)$. For each e_α, we can choose $y_\alpha\in e_\alpha$ not on S_α. Let

$$A = Z\backslash\bigcup_\alpha\{y_\alpha\} \text{ and } B = Z\backslash X$$

Then, A and B (a bunch of rectangles) are open and path connected. A deformation retracts onto X. B is contractible. Thus, $\pi_1(A)\simeq\pi_1(X)$. $A\cap B$ deformation retracts onto

$$\bigvee_\alpha \gamma_\alpha\cdot\varphi_\alpha\cdot\overline{\gamma_\alpha}$$

where $\varphi_\alpha = \partial e_\alpha$ and so, $\pi_1(A\cap B) = \langle\gamma_\alpha\cdot\varphi_\alpha\cdot\overline{\gamma_\alpha} : \alpha\in C\rangle$. By VKT, $\Phi:\pi_1(A)*\pi_1(B)\longrightarrow\pi_1(Z)$ with $\ker\Phi = \left\langle i_A(w)\, i_B(w)^{-1} : w\in\pi_1(A\cap B)\right\rangle$. Note that $i_B(w)^{-1}\simeq e$ because $\pi_1(B)=\{[e]\}$ and $i_A:\pi_1(A\cap B)\longrightarrow\pi_1(A)$ and $i_B:\pi_1(A\cap B)\longrightarrow\pi_1(A)$. Thus, $\ker\Phi = \langle i_A(w)\rangle = N$ and so, $i_*:\pi_1(A)\longrightarrow\pi_1(Z)$ is surjection with kernel N and so, $i_*:\pi_1(X)\longrightarrow\pi_1(X)$ is a surjection with kernel N.

For 2, Use the same decomposition as before. Observe that $A\cap B$ is a collection of punctured discs and so, deformation retracts to

$$\bigvee_\alpha S^{n-1}$$

which has a trivial fundamental group and hence simply connected. By VKT, $i_*:\pi_1(A)*\pi_1(B)\longrightarrow\pi_1(Z)$ is an isomorphism because the intersection has a trivial fundamental group $(\pi_1(A)=X,\ \pi_1(B)=\{[e]\},\pi_1(X)\simeq\pi_1(Y))$

For 3, use induction. ■