Orthomodularity and the incompatibility of relativity and quantum mechanics

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Abstract: We show that orthomodularity in general and non-existence of isotropic vectors in particular decisively yield the geometry of quantum mechanics and that a fundamental reason why quantum mechanics and relativity cannot be unified is because of the non-existence of isotropic vectors.

Completeness

- Completeness for Einstein = idea of pure state
- Completeness of Bohr = compatible observables
- $\Longrightarrow \omega(A) = Tr(\rho A) = \langle \mathbf{x}, A\mathbf{x} \rangle$
- $\nexists \omega : B(\mathcal{H}) \longrightarrow \mathbb{K}$ such that $\omega(AB) = \omega(A)\,\omega(B)$

I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more. After all, Hilbert space was obtained by generalising Euclidean space, footing on the principle of 'conserving the validity of all formal rules'. Now we begin to believe that it is not the vectors which matter, but the lattice of all linear (closed) subspaces. Because: 1) The vectors ought to represent the physical states, but they do it redundantly, up to a complex factor, only 2) and besides, the states are merely a derived notion, the primitive (phenomenologically given) notion being the qualities which correspond to the linear closed subspaces [3].

Issues

- $\frac{(0,1)+(1,0)}{\sqrt{2}} \equiv \frac{(0,1)-i(1,0)}{\sqrt{2}}$
- Why $\mathbb{C}$?[4][3]
- Why linear operators when measurement is non-linear?[1]
- Why separable?[2] (uncountable eigenvectors)
- Why associative law? [5]
- Hilbert spaces vs Semi-norm spaces

Issues

- $\mathbf{x} \neq 0$ is said to be isotropic if $\langle \mathbf{x}, \mathbf{x} \rangle = 0$
- Minkowski product for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^4$ is $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_2 + x_2 y_2 + x_3 y_3 - x_4 y_4$

Outline

- Our focus: skew fields $\mathbb{K}$ and seminorm spaces

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Outcomes: Isotropic vectors are important

Outcomes: No non-Archimedean fields for Quantum Mechanics

Outcomes: Multivalued operators are forced to be single-valued

Mappings

Definition

Let X be a vector space over $\mathbb{F}$ and Y be vector space over $\mathbb{K}$ and let $\phi : \mathbb{F} \longrightarrow \mathbb{K}$ be a homomorphism. Then, an operator $T : X \longrightarrow Y$ is a **ϕ-vector space homomorphism** between X and Y if for all $\mathbf{x}, \mathbf{y} \in X$ and scalars $\alpha \in \mathbb{F}$, $T(\alpha\mathbf{x} + \beta\mathbf{y}) = \phi(\alpha)\, T(\mathbf{x}) + \phi(\beta)\, T(\mathbf{y})$. T is an **isomorphism** if T and ϕ are bijective. A **ϕ-algebra homomorphism** is of the form $T((\alpha\mathbf{x})(\beta\mathbf{y})) = T(\alpha\beta\mathbf{x}\mathbf{y}) = \phi(\alpha\beta)\, T(\mathbf{x})\, T(\mathbf{y})$, which we shall call an **isomorphism** if ϕ and T are bijective.

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Definition

$T = \{(\mathbf{x}, \mathbf{z}) : \mathbf{x} \in V, \mathbf{z} \in W\}$ is a relation, then $(\alpha \mathbf{x} + \beta \mathbf{y}) Tz = \phi(\alpha) \mathbf{x} T \mathbf{z} + \phi(\beta) \mathbf{y} T \mathbf{z}$.

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Lemma

Preservation of multiplicative linear independence if T is injective (not ϕ)

Axioms for seminorm space N

- $\boxed{\|\mathbf{x}\| = 0 \implies \mathbf{x} = 0 \text{ (non-degeneracy)}}$
- $\|\alpha\mathbf{x}\| = |\alpha|\,\|\mathbf{x}\|$ for all $\alpha \in \mathbb{K}$, $\forall\mathbf{x} \in N$ (homogeneity)
- $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for arbitrary $\mathbf{x}, \mathbf{y} \in N$ or $\|\mathbf{x}+\mathbf{y}\| \leq \max\left(\|\mathbf{x}\|, \|\mathbf{y}\|\right)$
- Seminorm from underlying field: $\|\mathbf{x}\| := |g(\mathbf{x})|$
- Outcomes: $\|\mathbf{0}\| = 0$, $\|\mathbf{x}\| = \|-\mathbf{x}\|$ and $\|\mathbf{x}\| \geq 0$
- Norm: N/W where W =set $\mathbf{v}$ s.t. $\|\mathbf{v}\| = 0$
- $\left\|\mathbf{x}^2\right\| = \|\mathbf{x}\|^2 \implies \|\mathbf{xy}\| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|$[4] $\implies \|\mathbf{e}\| \geq 1$

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- Axiom of choice!

Sesquilinear forms

Definition

Let X be a vector space over $\mathbb{K}$. A f-**sesquilinear 2-form** is a function $\varphi : X \times X \longrightarrow \mathbb{K}$ such that $\forall \alpha \in \mathbb{K}$ and $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X$

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- $\varphi(\mathbf{x}, \mathbf{y} + \mathbf{z}) = \varphi(\mathbf{x}, \mathbf{y}) + \varphi(\mathbf{x}, \mathbf{z})$

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- $\varphi(\alpha\mathbf{x}, \mathbf{y}) = f(\alpha)\,\varphi(\mathbf{x}, \mathbf{y})$ where $f : \mathbb{K} \longrightarrow \mathbb{K}$ is an involutive anti-automorphism.

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- Outcomes: $\boxed{\varphi(\mathbf{0}, \mathbf{y}) = \varphi(\mathbf{x}, \mathbf{0}) = 0}$, $\boxed{\text{char}\mathbb{K} = 2 \text{ implies } \varphi(\mathbf{v}, \mathbf{v}) = 0 \iff \varphi(\mathbf{v}, \mathbf{w}) = -\varphi(\mathbf{w}, \mathbf{v})}$, $\boxed{\varphi(\mathbf{x}, \mathbf{y}) = f(\varphi(\mathbf{y}, \mathbf{x})) \iff \varphi(\mathbf{x}, \mathbf{x}) \in \mathbb{R}}$ and $\boxed{\varphi(\mathbf{x}, \mathbf{x})\varphi(\mathbf{y}, \mathbf{y}) \geq \varphi(\mathbf{x}, \mathbf{y})\varphi(\mathbf{y}, \mathbf{x}) \text{ if } \varphi(\mathbf{x}, \mathbf{x}) \geq 0}$

Sesquilinear forms

Lemma

$\varphi(\mathbf{x},\mathbf{x}) := \|\mathbf{x}\|^2$ *for* φ=*Hermitian and* $|f(\alpha)| = |\alpha|$

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Lemma

$\varphi(\mathbf{x},\mathbf{x}) := \|\mathbf{x}\|^2$ *for* φ*=Hermitian and* $|f(\alpha)| = |\alpha|$

Proof.

$\|\alpha\mathbf{x}\|^2 = |\alpha|^2 \|\mathbf{x}\|^2$ and $\|\mathbf{x}+\mathbf{y}\|^2$
$\leq |\varphi(\mathbf{x},\mathbf{x})| + |\varphi(\mathbf{x},\mathbf{y})| + |\varphi(\mathbf{y},\mathbf{x})| + |\varphi(\mathbf{y},\mathbf{y})| \leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2$
$\leq \max(|\varphi(\mathbf{x},\mathbf{x})|, |\varphi(\mathbf{x},\mathbf{y})|, |\varphi(\mathbf{y},\mathbf{x})|, |\varphi(\mathbf{y},\mathbf{y})|)$. Now, if
$\varphi(\mathbf{x},\mathbf{y}) = a + b$ for $a, b \in \mathbb{K}$ for $f(a) = a$ and $f(b) \neq b$, then
$|a|,|b| \leq \|\mathbf{x}\| \|\mathbf{y}\|$ [1] $\implies \max\{|a|, |b|\} \leq \|\mathbf{x}\| \|\mathbf{y}\|$ so that
$\max(|\varphi(\mathbf{x},\mathbf{x})|, |\varphi(\mathbf{y},\mathbf{x})|, |\varphi(\mathbf{y},\mathbf{y})|) = \max\{\|\mathbf{x}\|, \|\mathbf{y}\|\}$
If $\mathbf{x} \neq 0$ implies $\varphi(\mathbf{x},\mathbf{x}) > 0$, then N1 □

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$\varphi(\mathbf{x},\mathbf{x}) := \|\mathbf{x}\|^2$ *for* φ*=Hermitian and* $|f(\alpha)| = |\alpha|$

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$\varphi(\mathbf{x},\mathbf{y}) = a + b$ for $a, b \in \mathbb{K}$ for $f(a) = a$ and $f(b) \neq b$, then
$|a|,|b| \leq \|\mathbf{x}\|\|\mathbf{y}\|$ [1] $\implies \max\{|a|, |b|\} \leq \|\mathbf{x}\|\|\mathbf{y}\|$ so that
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If $\mathbf{x} \neq 0$ implies $\varphi(\mathbf{x},\mathbf{x}) > 0$, then N1 □

- $|\varphi(\mathbf{x},\mathbf{y})| \leq m\|\mathbf{x}\|\|\mathbf{y}\| \implies \varphi(\mathbf{x}_n,\mathbf{y}_n) \longrightarrow \varphi(\mathbf{x},\mathbf{y})$

Closed subspaces and associated algebra[2]

- $A \longmapsto A^{\perp\perp} \implies A \subseteq A^{\perp\perp}$, $A \subseteq B \implies A^{\perp\perp} \subseteq B^{\perp\perp}$ and $A^{\perp\perp\perp\perp} = A^{\perp\perp}$

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Theorem

$A^{\perp\perp}$ is the smallest subspace containing A

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Proof.

Assume there exists a closed B such that $A \subset B \subseteq A^{\perp\perp}$. Then, $B = B^{\perp\perp}$ and $A \subset B^{\perp\perp} \subseteq A^{\perp\perp}$ so that $B^{\perp} \subset A^{\perp}$ and $A^{\perp\perp\perp} = A^{\perp} \subseteq B^{\perp}$ and hence $B^{\perp\perp} = A^{\perp\perp}$. □

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Theorem

A closed relation ($T = T^{\perp\perp}$) T is linear

Closed subspaces and associated algebra[2]

Proof.

T is a subspace of $X \oplus X$. Plus $T(\alpha \mathbf{x}) = f(\alpha) T(\mathbf{x})$ if $\alpha(\mathbf{x}, \mathbf{y}) := (\alpha \mathbf{x}, f(\alpha) \mathbf{y})$ □

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- $T + S = \{(\mathbf{x}, \mathbf{y}) : \mathbf{y} = \mathbf{s} + \mathbf{t} \text{ for } \mathbf{s} \in \mathcal{R}(S), \mathbf{t} \in \mathcal{R}(T)\}$

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- $(S + T)R \subseteq SR + TR$. Converse holds if R is single-valued.

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- $(S + T)R \subseteq SR + TR$. Converse holds if R is single-valued.
- $\ker T = \ker S$ and $\mathcal{R}(S) = \mathcal{R}(T)$, then $S \subset T$ implies $S = T$.

Closed subspaces and associated algebra[2]

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A single-valued, linear adjoint of T will always exist

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Proof.

$U : X \times X \longrightarrow X \times X$ by $U(\mathbf{x}, \mathbf{y}) := (-\mathbf{y}, \mathbf{x})$. Well-defined+bijective.
$\Phi(\mathbf{z}, \mathbf{w}) := (\varphi \oplus \varphi)(\mathbf{z}, \mathbf{w})_{X \times X} = \varphi(\mathbf{z}_1, \mathbf{w}_1) + \varphi(\mathbf{z}_2, \mathbf{w}_2)$
$\implies \Phi(U(\mathbf{z}), \mathbf{w}) = \Phi\left(\mathbf{z}, U^{-1}(\mathbf{w})\right)$
For $M \subseteq X \times X$, $T^* = U\left(M^{\perp}\right) = U(M)^{\perp}$
$\varphi(T\mathbf{x}, \mathbf{y}) = \varphi(\mathbf{x}, T^*\mathbf{w})$ for $(\mathbf{x}, \mathbf{z}) \in T$ and $(\mathbf{y}, \mathbf{w}) \in T^*$ □

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Outcomes $\ker T^* = \mathcal{R}(T)^{\perp}$, $(\lambda T)^* = f(\lambda)\, T^*$, $\left(T^{-1}\right)^* = (T^*)^{-1}$,
$T^* = \left(-T^{-1}\right)^{\perp}$, $T^* = \left(-T^{-1}\right)^{\perp}$

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$\varphi(T\mathbf{x}, \mathbf{y}) = \varphi(\mathbf{x}, T^*\mathbf{w})$ for $(\mathbf{x}, \mathbf{z}) \in T$ and $(\mathbf{y}, \mathbf{w}) \in T^*$ $\square$

Outcomes $\ker T^* = \mathcal{R}(T)^{\perp}$, $(\lambda T)^* = f(\lambda) T^*$, $\left(T^{-1}\right)^* = (T^*)^{-1}$,
$T^* = \left(-T^{-1}\right)^{\perp}$, $T^* = \left(-T^{-1}\right)^{\perp}$

Outcomes $\mathcal{D}(T)^{\perp\perp} = E \iff T^*$ is single-valued

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- $\|T\| = \sup\limits_{\|\mathbf{x}\|\neq 0} \frac{\|T\mathbf{x}\|}{\|\mathbf{x}\|} \implies \|RT\| \leq \|R\|\,\|T\|$ (care for $\|\alpha T\| = |\phi(\alpha)|\,\|T\|$)

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Theorem

A unital Weak Banach algebra $(X, \|.\|)$ *is a complete subalgebra of* $B_\phi(X)$

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$L_\mathbf{x}(\mathbf{y}) := \mathbf{xy}$. Then, $L_\mathbf{x} \in B_\phi(X)$. Then, $L : X \longrightarrow B_\phi(X)$ as $L(\mathbf{x}) = L_\mathbf{x}$ is a homomorphism and $\|\mathbf{x}\|_o := \|L_\mathbf{x}\|$ is equivalent to $\|.\|$ □

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Theorem

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- Proof: $\forall \lambda \in \mathbb{K}$, $\lambda I \in B_\phi(X) \implies g(I) = e$. Consider orthogonal projection operators P and $Q \in B_\phi(X)$ s.t. $\dim P(X) = \dim Q(X)$. Then, $T : P(X) \longrightarrow Q(X)$, a partial isometry such that $P = T^*T$, $Q = TT^*$ so that $PQ = 0 \implies g(Q) = g(P) = 0$. Further, $P + Q = I \implies e = g(I) = g(P) + g(Q) = 0$

Riesz Representation Theorem on Hermitian Spaces[2]

Theorem

$\exists$ *cts linear functional* $g : (X, \varphi, \mathbb{K}) \longrightarrow X^*$ *such that* $\mathcal{R}(g) = X'$.

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Proof.

$g_{\mathbf{y}} : X \longrightarrow X^*$ s.t. $g_{\mathbf{y}}(\mathbf{x}) = \varphi(\mathbf{y}, \mathbf{x})$
(injective+well-define) $\Longrightarrow \mathcal{R}(g) \subseteq X'$. g_y cts since
$\ker g_{\mathbf{y}} = \{k\mathbf{y} : k \in \mathbb{K}\}^{\perp}$
Conversely, for $h \in X'$, $h = 0 \implies g_0 = h \implies h \in \mathcal{R}(g)$.
$h \neq 0 \implies \dim h = 1$
$\implies X = \ker h \oplus \{k\mathbf{v} : k \in \mathbb{K}\}$. Letting $w = f^{-1}\left(\varphi(\mathbf{v}, \mathbf{z})^{-1} h(\mathbf{v})\right)\mathbf{z}$
for $\mathbf{z} \in \ker h^{\perp}$ and $\mathbf{z} \notin \{k\mathbf{v} : k \in \mathbb{K}\}^{\perp}$ gives us $h(\mathbf{v}) = \varphi(\mathbf{v}, \mathbf{w})$.
$X \ni \mathbf{x} = \mathbf{x}_1 + \alpha v \implies h(\mathbf{x}) = \alpha h(\mathbf{v}) \implies \varphi(\mathbf{x}, \mathbf{w}) = \alpha\varphi(\mathbf{v}, \mathbf{w}) \implies$
$h = g_w$ □

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Proof.

If $\mathbf{y}$ is anisotropic, then $y \notin \{k\mathbf{y} : k \in \mathbb{K}\}^{\perp}$ so
$\ker g_{\mathbf{y}} = \{k\mathbf{y} : k \in \mathbb{K}\}^{\perp} \implies X = \ker g_{\mathbf{y}} \oplus \ker g_{\mathbf{y}}^{\perp}$ □

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Proof.

If $0 \neq \mathbf{y} \in X$ such that $\varphi(\mathbf{y}, \mathbf{y}) = 0$, then
$\{k\mathbf{y} : k \in \mathbb{K}\} \oplus \{k\mathbf{y} : k \in \mathbb{K}\}^{\perp} \subset X$ □

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X is orthomodular $\iff L{=}C(X)$ is orthomodular

- If a Hermitian space is orthomodular, then $\langle F \rangle = F^{\perp\perp}$ and such sets form atomic ortholattice which is isomorphic to the lattice of closed subspaces of a Hilbert space over an arbitrary Archimedean skew field[6].

Solr's theorem

Theorem

Let $(X, \mathbb{K}, \varphi)$ *be an infinite dimensional orthomodular space over a skew field* $\mathbb{K}$ *which contains an orthonormal system* $(e_i)_{i \in \mathbb{N}}$. *Then* $\mathbb{K}$ *is either* $\mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{H}$ *and* $(X, \mathbb{K}, \varphi)$ *is a Hilbert space [4]*

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Theorem

Let $(X, \mathbb{K}, \varphi)$ be an infinite dimensional orthomodular space over a skew field $\mathbb{K}$ which contains an orthonormal system $(e_i)_{i\in\mathbb{N}}$. Then $\mathbb{K}$ is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and $(X, \mathbb{K}, \varphi)$ is a Hilbert space [4]

Proof.

$n\mathbf{x} = \langle \sum_{i=0}^{n} e_i \rangle \mathbf{x} = 0 \iff \langle \sum_{i=0}^{n} e_i \rangle = 0 \iff n = 0$
$\implies \mathbb{Q} \subset \mathbb{K}$
$\forall (\alpha_i)_{i\in\mathbb{N}^*} \in \mathbb{Q}^{\mathbb{N}^*}$ with $\alpha := \sum_{i=0}^{\infty} \alpha_i^2 \in \mathbb{Q}$, then $\exists x = \sum_{i\in\mathbb{N}^*} \alpha_i e_i \in X$, with $\langle x \rangle = \alpha$
Define $\sum_{i=0}^{\infty} \alpha_i^2 \longmapsto \langle \sum_{i\in\mathbb{N}} \alpha_i e_i \rangle$
This is multiplicative linear function so that $\mathbb{R} \subset \mathbb{K}$
$\implies (\alpha_i)_{i\in\mathbb{N}} \in l_2(\mathbb{R})$ with $\alpha := \sum_{i=0}^{\infty} \alpha_i^2$, $\exists \mathbf{x} = \sum_{i\in\mathbb{N}} \alpha_i e_i \in X$ such that $\langle a \rangle = \alpha$ □

Solr's theorem

Proof.

(cotd.)

Next, $\mathbb{R} \subset Z = \{x \mid xy = yx, \forall y \in \mathbb{K}\} \implies \mathbb{R} = S(\mathbb{K})$ using

$S \subseteq P := \left\{ \langle x \rangle \mid 0 \neq x = \sum_{i \in \mathbb{N}} \xi_i e_i,\ \xi_i \in \mathbb{R}(\gamma) \forall i \in \mathbb{N} \text{ and } \langle x \rangle \in \mathbb{R}(\gamma) \right\}$

where $\gamma \in S$

$\lambda \in \mathbb{K} \setminus \mathbb{R} \implies \mathbb{R}(\lambda) \cong \mathbb{C}$

$\lambda \in \mathbb{K} \setminus \mathbb{C} \implies \mathbb{C} + \mathbb{C}\lambda \cong \mathbb{H} \implies$

$\lambda \in \mathbb{K} \setminus \mathbb{H} \implies \mathbb{H} + \mathbb{H}\lambda \cong \mathbb{H}$, contradiction

Hence $X \cong l_2(\mathbb{K})$ and $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ □

Conclusion

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- $\impliedby$ Non-existence of isotropic vectors

Future Work

- Over which non-Archimedean fields are Hermitian spaces orthomodular?

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- Does there exist a (countable?) eigenbasis decomposition of a non-linear operator on a Hermitian space over a non-Archimedean field?

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